M 191 Lect #18 4-1-09

Suppose we have a function whose table values are given below.

	x	$y = f(x)$	
	2.0	7	y_0
	2.1	6	
	2.2	6	
Δx	2.3	5	y_1
	2.4	5	
	2.5	5	
	2.6	4	y_2
		4	
		4	
Δx	2.9	3	y_3
	3.2	6	y_4
Δx	3.5	8	y_5
	3.8	10	y_6
Δx	4.1	13	y_7
	4.4	12	y_8
	4.7	15	y_9
	⋮		

We wish to estimate the area under the curve $y = f(x)$ defined for $[a,b] = [2.0, 4.4]$ using $n = 4$ rectangles using these estimates

R_4 right hand rule

L_4. left hand rule

M_4 midpoint rule

U_4 upper rectangles

Low_4 lower rectangles

$$A \approx R_4 = \Delta x\left(y_2 + y_4 + y_6 + y_8\right)$$
$$= .6\left(4 + 6 + 10 + 12\right) = .6(32)$$
$$= 19.2$$

$$A \approx L_4 = \Delta x\left(y_0 + y_2 + y_4 + y_6\right)$$
$$= .6\left(7 + 4 + 6 + 10\right) = .6(27)$$
$$= 16.2$$

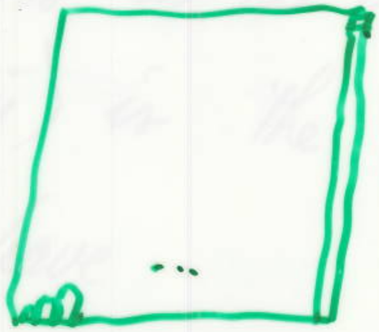
$$A \approx M_4 = \Delta x\left(y_1 + y_3 + y_5 + y_7\right)$$
$$= .6\left(5 + 5 + 8 + 13\right) = .6(31)$$
$$= 18.6$$

$$A \approx U_4 = \Delta x\left(y_0 + y_4 + y_6 + y_7\right)$$
$$= .6\left(7 + 6 + 10 + 13\right) = .6(36)$$
$$= 21.6$$

$$A \approx Low_4 = \Delta x\left(y_2 + y_2 + y_4 + y_6\right)$$
$$= .6\left(4 + 4 + 6 + 10\right) = .6(24)$$
$$= 14.4$$

$1+2+3+4+5+6+\cdots\cdots+100$

7



101

100

$7\cdot 8 = 56$

$\frac{56}{2} = 28$

$\sum_{k=1}^{7} k = 28$

$100(101) = 10100$

$\frac{10100}{2} = 5050$

$\sum_{k=1}^{100} k$

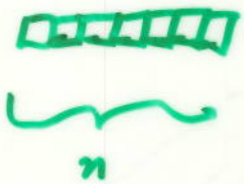
$1+2+\cdots\cdots+n$

$$\frac{n(n+1)}{2} = \sum_{k=1}^{n} k$$

$$\boxed{\sum_{k=1}^{n} k = \frac{n(n+1)}{2}}$$

$$\boxed{\sum_{k=1}^{n} 1 = n}$$

$$\frac{n}{1}$$



$$1^2 + 2^2 + 3^2 + 4^2 + \cdots + n^2$$

$$\boxed{\sum_{k=1}^{n} k^2 = \frac{n(n+1)(2n+1)}{6}}$$

$$\boxed{\sum_{k=1}^{n} k^3 = \frac{[n(n+1)]^2}{4}}$$

Now we return to the fcn

$y = f(x) = x^2+1$ and the interval

$[a,b] = [0, 2]$ and use n rectangles

We'll then take a limit as $n \to \infty$.

$y = f(x) = x^2+1$

$$R_n = U_n = \Delta x \left(\sum_{k=1}^{n} f(0 + k\Delta x) \right)$$

$(a+b)^2$
$a^2+2ab+b^2$

$$= \Delta x \left(\sum_{k=1}^{n} \left[(0 + k\Delta x)^2 + 1 \right] \right)$$

$$= \Delta x \left(\sum_{k=1}^{n} \left(\cancel{0^2}^{\,0} + \cancel{2 \cdot 0 \cdot k\Delta x}^{\,0} + k^2\Delta x^2 + 1 \right) \right)$$

$$= \Delta x \sum_{k=1}^{n} \left(k^2 \Delta x^2 + 1 \right)$$

$$\boxed{\Delta x = \frac{b-a}{n} = \frac{2-0}{n} = \frac{2}{n}}$$

$$= \frac{2}{n} \sum_{k=1}^{n} \left(k^2 \frac{4}{n^2} + 1 \right)$$

$$R_n = \frac{2}{n}\left[\sum_{k=1}^{n} k^2 \frac{4}{n^2} + \sum_{k=1}^{n} 1\right]$$

$$= \frac{2}{n}\left[\frac{4}{n^2}\boxed{\sum_{k=1}^{n} k^2} + \boxed{\sum_{k=1}^{n} 1}\right]$$

$$= \frac{2}{n}\left[\frac{4}{n^{\cancel{2}}} \cdot \frac{\cancel{n}(n+1)(2n+1)}{6} + n\right]$$

$$= \frac{2}{n}\left[\frac{\overset{2}{\cancel{4}}(n+1)(2n+1)}{n \cdot \cancel{6}\,3} + n\right]$$

$$R_n = \frac{4(n+1)(2n+1)}{n \cdot n \cdot 3} + 2\frac{\cancel{n}}{\cancel{n}}$$

exact area

$$= A = \lim_{n\to\infty} R_n = \lim_{n\to\infty} \frac{4(n+1)(2n+1)}{n \cdot n \cdot 3} + \lim_{n\to\infty} 2$$

$$= \frac{4}{3} \cdot \lim_{n\to\infty} \frac{n+1}{n} \cdot \lim_{n\to\infty} \frac{2n+1}{n} + 2$$

$$= \frac{4}{3} \cdot 1 \cdot 2 + 2 = \frac{8}{3} + 2 = 2\tfrac{2}{3} + 2 = 4\tfrac{2}{3}$$

So we have just found a long horrible arduous lengthy way to arrive at the exact area under the curve of a simple easy function $y = x^2 + 1$ on a really easy interval $[0, 2]$

We wish earnestly for an easier way.

The Fundamental Theorem of Calculus tells us how to do this an easier way.

First some notation:

$$A = \lim_{n \to \infty} R_n$$

$$\int_a^b f(x)\,dx = \underbrace{\lim_{n \to \infty} \sum_{k=1}^{n} f(a + k\Delta x) \cdot \Delta x}_{\text{"lim sum"}}$$

FTC 2

$$\int_a^b f(x)\,dx = \int f(x)\,dx \Big]_a^b$$

$$= F(x) \Big]_a^b$$

the definite integral of f w/r x from a to b

$$= F(b) - F(a)$$

$$A = \int_0^2 (x^2+1)\,dx = \int (x^2+1)\,dx \Big]_0^2$$

$$= \frac{x^3}{3} + x + C \Big]_{x=0}^{x=2}$$

$$= \left(\frac{8}{3} + 2 + C\right) - \left(\frac{0}{3} + 0 + C\right)$$

$$= \frac{8}{3} + 2 = \frac{8}{3} + \frac{6}{3} = \frac{14}{3} = 4\frac{2}{3}$$

the exact area under the curve (in four lines)