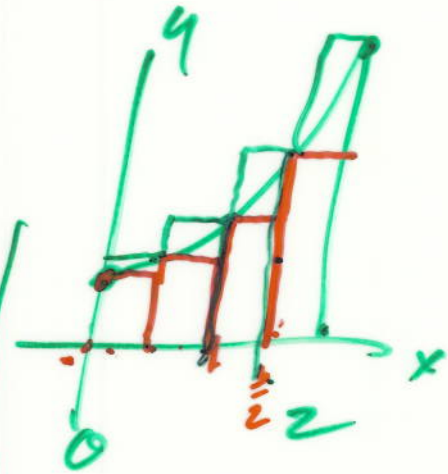
M 191 Lect #19 4-4-11

Continuing to estimate the area under $y = x^2 + 1$ and above the interval $[0, 2]$ $y = f(x)$



$Low_4 \le A \le U_4$

$\frac{1}{2} \cdot 1 + \frac{1}{2} \cdot \frac{5}{4} + \frac{1}{2} \cdot 2 + \frac{1}{2} \cdot \frac{13}{4}$

$\frac{1}{2}\left(\frac{4}{4} + \frac{5}{4} + \frac{8}{4} + \frac{13}{4}\right)$

$y = \left(\frac{3}{2}\right)^2 + 1$

$\frac{1}{2} \cdot \frac{30}{4}$ $\quad \frac{1}{2} \cdot \frac{5}{4} + \frac{1}{2} \cdot 2 + \frac{1}{2} \cdot \frac{13}{4} + \frac{1}{2} \cdot 5$ $= \frac{9}{4} + \frac{4}{4} = \frac{13}{4}$

$\frac{1}{2}\left(\frac{9}{2} + 7\right) = \frac{1}{2}\left(\frac{23}{2}\right) = \frac{23}{4}$

$\frac{15}{4}$

$3.75 \le A_4 \le 5.75$ Guess $A_4 = 4.75$

We could redo this for A_8, then A_{16}, and when our required decimal began to repeat, then stop and say that's the area.

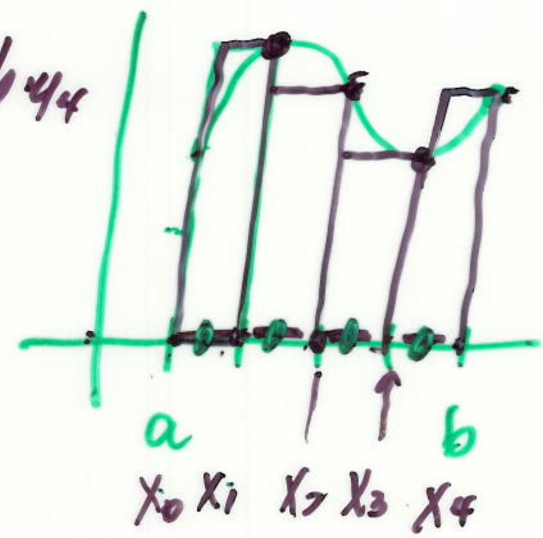
What will the formulas for A_4 look like. This time we use L_4 for left endpoints & R_4 for right endpoints & M_4 for midpoint rules.

$y = f(x)$ on interval $[a, b]$

$$R_4 = \text{base} \cdot y_1 + b \cdot y_2 + b \cdot y_3 + b \cdot y_4$$

$$= \Delta x \cdot f(x_1) + \Delta x\, f(x_2) + \Delta x\, f(x_3) + \Delta x\, f(x_4)$$

$$= \Delta x \cdot f(a + 1\Delta x) + \Delta x \cdot f(a + 2\Delta x) + \Delta x \cdot f(a + 3\Delta x) + \Delta x \cdot f(a + 4\Delta x)$$



$$\Delta x = \frac{b-a}{4} = \frac{b-a}{n}$$

$x_0 = a + 0 \cdot \Delta x$

$x_1 = a + 1 \cdot \Delta x$

$x_2 = a + 2\Delta x$

$x_3 = a + 3\Delta x$

$x_4 = a + 4\Delta x$

$$R_4 = \sum_{k=1}^{4} \Delta x \cdot f(a + k\Delta x)$$

$$R_4 = \sum_{k=1}^{4} f(a + k\Delta x) \cdot \Delta x$$

$$L_4 = \sum_{k=1}^{4} f(a + (k-1)\Delta x) \cdot \Delta x$$

$$M_4 = \sum_{k=1}^{4} f(a + (k-\tfrac{1}{2})\Delta x) \Delta x$$

For the 3 rules, using $y=f(x)$, $[a,b]$ and n rectangles

$$\Delta x = \frac{b-a}{n}$$

$$R_n = \sum_{k=1}^{n} f(a+k\Delta x)\cdot\Delta x$$

$$L_n = \sum_{k=1}^{n} f(a+(k-1)\Delta x)\cdot\Delta x$$

$$M_n = \sum_{k=1}^{n} f\left(a+\left(k-\tfrac{1}{2}\right)\Delta x\right)\cdot\Delta x$$

$$A_{exact} = \lim_{n\to\infty} R_n = \underbrace{\lim_{n\to\infty} \sum_{k=1}^{n} f(a+k\Delta x)\cdot\Delta x}_{\text{"lim sum"}}$$

The name of this "lim sum" definite integral of f from a to b

$$\int_a^b f(x)\,dx = \lim_{n\to\infty} \sum_{k=1}^{n} f(a+k\Delta x)\cdot\Delta x$$

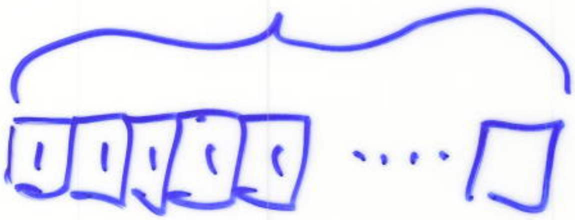
We need a few sigma formulas to demonstrate the long way of evaluating (definite) integrals

$$\sum_{k=1}^{n} 1 = \overbrace{1+1+1+\cdots+1}^{n} = n = \frac{n}{1}$$

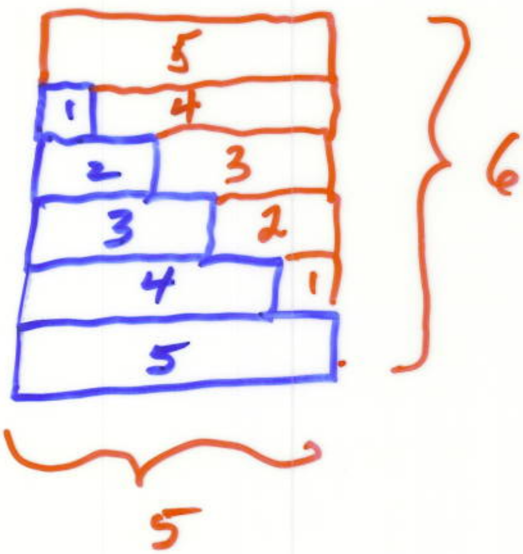
$$\sum_{k=1}^{n} k = 1+2+3+4+5+\cdots+n = \frac{n(n+1)}{2}$$

$$\sum_{k=1}^{n} k^2 = 1+4+9+16+\cdots+n^2 = \frac{n(n+1)(2n+1)}{6}$$

$$\sum_{k=1}^{n} k^3 = 1+8+27+64+\cdots+n^3 = \left(\frac{n(n+1)}{2}\right)^2$$



$= n$



$$2 \sum_{k=1}^{5} k = 5 \cdot 6$$

$$\sum_{k=1}^{5} k = \frac{5 \cdot 6}{2}$$

$$\sum_{k=1}^{n} k = \frac{n(n+1)}{2}$$

Find the value of the integral

$\int_0^2 (x^2+1)\,dx$ which is the exact Area of our original problem tonight.

$\Delta x = \frac{2-0}{n} = \frac{2}{n}$

$$= \lim_{n\to\infty} \sum_{k=1}^{n} f\left(0 + k\cdot\frac{2}{n}\right)\cdot\frac{2}{n}$$

$$= \lim_{n\to\infty} \sum_{k=1}^{n} \left(\left(k\cdot\frac{2}{n}\right)^2 + 1\right)\cdot\frac{2}{n}$$

$$= \lim_{n\to\infty} \sum_{k=1}^{n} \frac{8\cdot k^2}{n^3} + \frac{2}{n}$$

$$= \lim_{n\to\infty} \left[\sum_{k=1}^{n} \frac{8}{n^3}\cdot k^2 + \sum_{k=1}^{n} \frac{2}{n}\right]$$

$$= \lim_{n\to\infty} \left[\frac{8}{n^3} \sum_{k=1}^{n} k^2 + \frac{2}{n} \sum_{k=1}^{n} 1\right]$$

$$= \lim_{n\to\infty}\left[\frac{8}{n^{\cancel{3}\,2}}\cdot\frac{\cancel{n}(n+1)(2n+1)}{6}+\frac{2}{\cancel{n}}\cdot\cancel{n}\right]$$

$$= \lim_{n\to\infty}\left[\frac{8}{6}\;\overset{1}{\cancel{\frac{n+1}{n}}}\;\overset{2}{\cancel{\frac{2n+1}{n}}}\right] + \lim_{n\to\infty} 2$$

$$= \frac{8}{6}\cdot 1\cdot 2 + 2$$

$$= \frac{8}{3} + \frac{6}{3} = \frac{14}{3} = 4\tfrac{2}{3} =$$

= value of the integral

= the exact area under that original curve.

Now evaluate the definite integral

$\int_0^2 (x^2+1)\,dx$ using the FTC 2

$$\int_0^2 (x^2+1)\,dx \xrightarrow{\text{antider}} = \frac{x^3}{3}+x+C\Big]_{x=0}^{2}$$

$$= \left(\frac{2^3}{3}+2+C\right) - \left(\frac{0^3}{3}+0+C\right)$$

$$= \frac{8}{3}+2 = \frac{8}{3}+\frac{6}{3} = \frac{14}{3} = 4\tfrac{2}{3}$$

	f
$\frac{x^3}{3}+x+C$ $\leftarrow$	x^2+1+0