M 192 Lect # 23 11-14-11

Here are the five operations that convert a fcn and its power series to another fcn and its power series

1. Replace x or $x-a$ by $\pm b(x \pm d)^p$

   p a whole #

2. Add (or Subtract) a const, a poly, or a series in x or $x-a$

3. Mply (& sometimes divide by) a const, a poly or a series in x or $x-a$

4. Differentiate

5. Integrate

<u>Ex</u> of ③ mpy by x^4

$$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n \qquad |x|<1$$

③ mpy by x^4

$$\frac{x^4}{1-x} = x^4 \sum_{n=0}^{\infty} x^n = \sum_{n=0}^{\infty} x^{n+4} \qquad |x|<1$$

$$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n \qquad |x|<1$$

② subtract $1+x$

$$\frac{1}{1-x} - (1+x)\frac{(1-x)}{(1-x)} = -1-x + \sum_{n=0}^{\infty} x^n \qquad |x|<1$$

$$\frac{1-(1-x^2)}{1-x} = \cancel{-1}\cancel{-x} + \cancel{x^0} + \cancel{x^1} + \sum_{n=2}^{\infty} x^n$$

$$\frac{x^2}{1-x} = \sum_{n=2}^{\infty} x^n \qquad |x|<1$$

$$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n \qquad |x| < 1$$

(4) Differentiate

$$-1\frac{1}{(1-x)^2}(-1) = \sum_{n=0}^{\infty} n x^{n-1} \qquad |x| < 1$$

or $n=1$

$$\frac{1}{(1-x)^2} = \sum_{n=1}^{\infty} n x^{n-1}$$

$$\frac{1}{(1-x)^2} = \sum_{n=0}^{\infty} (n+1) x^n \qquad |x| < 1$$

We now create a new tree top (series)

Given a fcn f(x) we want to create a polynomial $p_n(x)$ to be as much like the fcn as possible.

$$p_n(x) = c_0 + c_1 x + c_2 x^2 + c_3 x^3 + \cdots + c_n x^n$$

1. $f(0)$ to match $p_n(0)$ match the height @ $x=0$

   Plug 0 in for f & for p_n

$$f(0) = c_0$$

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2. match slopes of f & p_n at $x=0$

$$f'(0) = p_n'(0)$$

$$p_n'(x) = c_1 + 2c_2 x + 3c_3 x^2 + 4c_4 x^3 + \cdots + n c_n x^{n-1}$$

$$f'(0) = p_n'(0) = c_1$$

(3) match mood i.e. concavity @ $x=0$

$$f''(0) = p_n''(0)$$

$$p_n''(x) = 2c_2 + 2\cdot 3c_3x^1 + 3\cdot 4c_4x^2 + \dots$$

$$f''(0) = p_n''(0) = 2c_2 \qquad c_2 = \frac{f''(0)}{2}$$

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(4) match mood swing i.e. deriv of concavity @ $x=0$

$$f'''(0) = p_n'''(0)$$

$$p_n'''(x) = 1\cdot 2\cdot 3\, c_3x^0 + 2\cdot 3\cdot 4c_4x^1 + \dots$$

$$f'''(0) = p_n'''(0) = 1\cdot 2\cdot 3c_3 \qquad c_3 = \frac{f'''(0)}{1\cdot 2\cdot 3}$$

$$c_3 = \frac{f'''(0)}{3!} \qquad c_4 = \frac{f^{(4)}(0)}{4!}$$

$$\dots\ c_k = \frac{f^{(k)}(0)}{k!} \qquad c_n = \frac{f^{(n)}(0)}{n!}$$

The Taylor (MacLaurin if $a=0$) Polynomial for a fcn $f(x)$ at $x=a=0$ is

$$f(x) \approx p_n(x) = \sum_{k=0}^{n} c_k x^k$$

$$= \sum_{k=0}^{n} \frac{f^{(k)}(0)}{k!} x^k$$

If we choose to let $n \to \infty$ we get

$$f(x) = \lim_{n \to \infty} p_n(x) \qquad (x-a)^n$$

$$= \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n$$ This is called

the Taylor series for the fcn $f(x)$ expanded about $x=a=0$.

In the general case we choose to expand about $x = a$

$$\boxed{f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n}$$