Math 192 Lect #25 11-21-11

Now we learn to bound the remainder of a Taylor series

$$f(x) = e^x = \sum_{n=0}^{\infty} \frac{1}{n!} x^n$$

$$= \sum_{k=0}^{n} \frac{1}{k!} x^k + \sum_{k=n+1}^{\infty} \frac{1}{k!} x^k$$

$$= P_n(x) + R_n(x)$$

We wish to find a upper bound for $|R_n(x)|$ for in in a certain interval about 0 (in general about $x=a$)

A bound for the remainder is

$$|R_n(x)| = \frac{M}{(n+1)!}(x-a)^{n+1}$$ where

$$M = \max_{\substack{z\in[a,x] \\ \text{or} \\ z\in[x,a]}} |f^{(n+1)}(z)|$$

z between x and a

Let's return to

$f(x) = e^x$ fcn

and lets use $x = 1.8$

$a = 0$ in our present case, $P_n = P_4$ so

$n = 4,\ n+1 = 5$

k	$f^{(k)}(x)$	$f^{(k)}(0)$
0	e^x	$e^0 = 1$
1	e^x	$e^0 = 1$
2	e^x	$e^0 = 1$
3	e^x	$e^0 = 1$
4	e^x	$e^0 = 0$
5	e^x	e^z

$$f(x) \doteq P_4(x) = P_4(x)$$
$$= 1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \frac{x^4}{24}$$

① approx $f(1.8) \doteq P_4(1.8) = 1 + 1.8 + \frac{1.8^2}{2} + \frac{(1.8)^3}{6} + \frac{(1.8)^4}{24}$

$= 1 + 1.8 + 1.62 + .972 + .4374$

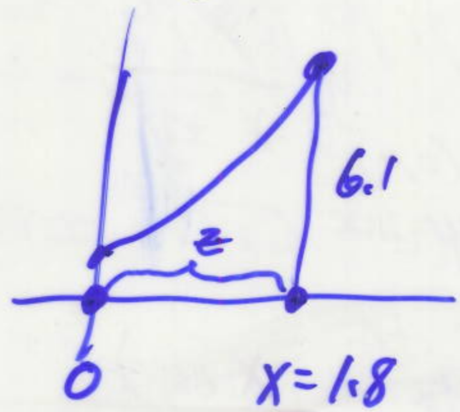
$= 5.8294$

Recall $e^{1.8} = 6.04$ from Calculator

(2) Find the formula for the bound for Remainder

$$|R_{\underset{4}{n}}(x)| \le \frac{M}{5!}(x-0)^5$$

$$f^{(5)}(z) = e^z$$

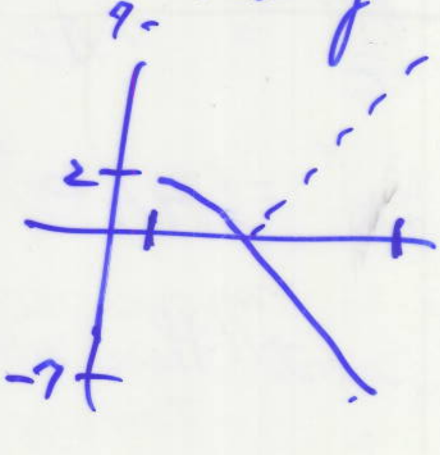


$$M = \max\left|f^{(5)}(z)\right| \le e^x = e^{1.8} \le 6.1$$

$$|R_4(\underset{1.8}{x})| \le \frac{6.1}{120}(1.8-0)^5$$

$$|R_4(x)| \le .97 \leftarrow \text{error bound}$$

What if



Actual error was $\approx .2$

So error bound is this time about 5 times as big as the actual error.

③ Write Taylor series

$$e^x = \sum_{n=0}^{\infty} \frac{1}{n!}(x-0)^n = \sum_{n=0}^{\infty} \frac{x^n}{n!}$$

The Binomial Theorem

$$(a+b)^0 = 1$$

$$(a+b)^1 = 1a + 1b$$

$$(a+b)^2 = 1a^2 + 2ab + 1b^2$$

$$(a+b)^3 = 1a^3 + 3a^2b + 3ab^2 + 1b^3$$

$$(a+b)^4 = 1a^4 + 4a^3b^1 + 6a^2b^2 + 4a^1b^3 + 1b^4$$

$$(a+b)^5 = 1a^5 + 5a^4b^1 + 10a^3b^2 + 10a^2b^3 + 5ab^4 + 1b^5$$

1

1 1

1 2 1

1 3 3 1

1 4 6 4 1

1 5 10 10 5 1

1 6 15 20 15 6 1

Replace a by 1 and b by x

$$(1+x)^5 = 1 + 5x + 10x^2 + 10x^3 + 5x^4 + 1x^5$$

$$\uparrow$$

$\binom{5}{2}$ The number of combinations of 5 things taken 2 at a time.

"5 choose 2"

$$\binom{5}{2} = \frac{5 \cdot \cancel{4}^{2}}{1 \cdot \cancel{2}} = 10$$

$$\binom{5}{2} \overset{def}{=} \frac{5!}{2!\,(5-2)!} = \frac{\cancel{1 \cdot 2 \cdot 3} \cdot 4 \cdot 5}{1 \cdot 2 \; \cancel{1 \cdot 2 \cdot 3}}$$

Find the x^3 term in $(1+x)^{20}$

$$\binom{20}{3} = \frac{\overset{10}{\cancel{20}} \cdot 19 \cdot \overset{6}{\cancel{18}}}{1 \cdot \cancel{2} \cdot \cancel{3}} = 1140$$

$$= \frac{20 \cdot 19 \cdot 18}{1 \cdot 2 \cdot 3}$$

Let's make a table to find the Taylor polynomial and Taylor series. for the binomial fcn $(1+x)^p$

k	$f^{(k)}(x)$	$f^{(k)}(a)$, $a = 0$
0	$(1+x)^p$	1
1	$p(1+x)^{p-1}$	p
2	$p(p-1)(1+x)^{p-2}$	$p(p-1)$
3	$p(p-1)(p-2)(1+x)^{p-3}$	$p(p-1)(p-2)$

polynomial $Q_5(x) = \frac{1}{0!}x^0 + \frac{p}{1!}x^1$

$$+ \frac{p(p-1)}{2!}x^2 + \frac{p(p-1)(p-2)}{3!}x^3$$

$$+ \underbrace{\frac{p(p-1)(p-2)(p-3)}{4!}}_{\binom{p}{4}}x^4 + \underbrace{\frac{p(p-1)(p-2)(p-3)(p-4)}{1\ 2\ 3\ 4\ 5}}_{\binom{p}{5}}x^5$$

So our Taylor polynomial for

$$(1+x)^p \approx Q_5(x)$$

$$= 1 + \frac{p}{1}x + \frac{p(p-1)}{2!}x^2 + \cdots$$

$$= \binom{p}{0}x^0 + \binom{p}{1}x^1 + \binom{p}{2}x^2$$

$$+ \binom{p}{3}x^3 + \binom{p}{4}x^4 + \binom{p}{5}x^5$$

$$= \sum_{n=0}^{5} \binom{p}{n} x^n$$

Taylor Series for binomial is

$$\boxed{(1+x)^p = \sum_{n=0}^{\infty} \binom{p}{n} x^n}$$

Get 4 terms (up through x^3)
for $(1+x)^{-1/2} = \frac{1}{\sqrt{1+x}}$

$$= \binom{-1/2}{0} + \binom{-1/2}{1} x + \binom{-1/2}{2} x^2$$

$$= 1 + \frac{-1/2}{1} x + \frac{-1/2(-1/2-1)}{1 \cdot 2} x^2 + \frac{-1/2(-1/2-1)(-1/2-2)}{1\ 2\ 3} x^3$$

$$= 1 - \tfrac{1}{2} x + \frac{-1/2\,(-3/2)}{1 \cdot 2} x^2 + \frac{-1/2(-3/2)(-5/2)}{1\ 2\ 3} x^3$$

$$= 1 - \tfrac{1}{2} x + \frac{3}{8} x^2 - \frac{15}{48}$$

$$1 - \tfrac{1}{2} x + \tfrac{3}{8} x^2 - \tfrac{5}{16} x^3 = Q_3(x)$$

Fact: $|x| < 1$