Steps for Completing the Linear Transformation Box

Given a vs V with 2 or 3 given bases and
given a vs W " " " " " " and
given a Transformation $T : V \to W$ in some form from which we can find how one of the bases of V is associated with a basis of W,
we wish to describe how to Transform any $v \in V$ to $T(\vec{v})$ in W and vice versa

{ (1) Rule (2) Basis (3) Matrix

1. Proclaim one basis of V to be the standard basis S of V,
   Name the other one or two bases of V B and D

2. Proclaim one basis of W to be the standard basis U
   Name the other one or two bases of W E and F

3. Since S is a basis, write $B_1 = lc$ S's. Name the coeff $[B_1]_S$, a column matrix
   $\vdots$
   Since S is a basis, write $B_n = lc$ S's. Name the coeff $[B_n]_S$ a column matrix

4. Form and name the augmented matrix $[\,[B_1]_S \vdots \cdots \vdots [B_n]_S\,] = Bcol$

5. Repeat steps 3 & 4 to form Dcol

6. Repeat steps 1,3,4,5 to form Ecol and Fcol

7. Define $Scol = I_n$ and $Ucol = I_m$

8. For each of the three bases of V define an isomorphism from V to $\mathbb{R}^n$
   For $v \in V$, $v = lc$ S's so name coeffs $[v]_S$
   $v = lc$ B's so name coeffs $[v]_B$
   $v = lc$ D's so name coeffs $[v]_D$

9. Repeat step 8 for $w \in W$

10. Define three isomorphisms
    $IsoS : V \to \mathbb{R}^n : v \longmapsto [v]_S$, i.e., $IsoS(\vec{v}) = [v]_S$
    $IsoB \quad \mathbb{R}^n \quad [v]_B \quad IsoB(\vec{v}) = [v]_B$
    $IsoS \quad \mathbb{R}^n \quad [v]_D \quad IsoD(\vec{v}) = [v]_D$

11. Repeat step 10 for $w \in W$

12. Note $\mathcal{B}_1 = lc\ \mathcal{S}'s$ or $\mathcal{B}_1 = [\![\mathcal{S}_1 \cdots \mathcal{S}_n]\!] \cdot [\mathcal{B}_1]_S$

$\vdots$

$\mathcal{B}_n = [\![\mathcal{S}_1 \cdots \mathcal{S}_n]\!] \cdot [\mathcal{B}_n]_S$

$$\text{So } [\![\mathcal{B}_1 \cdots \mathcal{B}_1]\!] = [\![\mathcal{S}_1 \cdots \mathcal{S}_n]\!] \left[[\mathcal{B}_1]_S \,\vdots \cdots \vdots\, [\mathcal{B}_n]_S \right] = [\![\mathcal{S}_1 \cdots \mathcal{S}_n]\!]\ Bcol$$

13. Repeat step 12 for $\mathcal{D}col$, $\mathcal{E}col$, $\mathcal{F}col$, $Scol = I_n$, $Ucol = I_m$

14. For each of the three bases, let $v \in V$, $w \in W$

$$v = lc\ \mathcal{S}'s = [\![\mathcal{S}_1 \cdots \mathcal{S}_n]\!] \cdot [v]_S = [\![\mathcal{S}_1 \cdots \mathcal{S}_n]\!]\ Scol\ [v]_S$$

$$[\![\mathcal{B}_1 \cdots \mathcal{B}_n]\!] \cdot [v]_B = [\![\mathcal{S}_1 \cdots \mathcal{S}_n]\!]\ Bcol\ [v]_B$$

$$[\![\mathcal{D}_1 \cdots \mathcal{D}_n]\!] \cdot [v]_D = [\![\mathcal{S}_1 \cdots \mathcal{S}_n]\!]\ \mathcal{D}col\ [v]_D$$

$$\text{So } [v]_S = Scol\ [v]_S = Bcol\ [v]_B = \mathcal{D}col\ [v]_D$$

15. Repeat step 14 to get

$$[w]_U = \underbrace{Ucol}_{I}\ [w]_U = Ecol\ [w]_E = Fcol\ [w]_F$$

16. Suppose $T : V \mapsto W : \vec{v} \mapsto T(v) = \vec{w}$ is defined rel to $\mathcal{S}$ and $\mathcal{U}$ bases

$\mathcal{S}_1 \mapsto T(\mathcal{S}_1) = [\![\mathcal{U}_1 \cdots \mathcal{U}_m]\!] \cdot [T(\mathcal{S}_1)]_U$

$\vdots$

$\mathcal{S}_n \mapsto \qquad [T(\mathcal{S}_n)]_U$

$\Big\}$ We form matrix $A = \left[[T(\mathcal{S}_1)]_E \,\vdots \cdots \vdots\, [T(\mathcal{S}_n)]_E \right]$

17.

is defined rel to $\mathcal{B}$ and $\mathcal{E}$ bases

$\mathcal{B}_1 \mapsto T(\mathcal{B}_1) = [\![\mathcal{E}_1 \cdots \mathcal{E}_m]\!] \cdot [T(\mathcal{B}_1)]_E$

$\vdots$

$\Big\}$ We form matrix $C = \left[[T(\mathcal{B}_1)]_E \,\vdots \cdots \vdots\, [T(\mathcal{B}_n)]_E \right]$

We form matrix $G = \left[[T(\mathcal{B}_1)]_F \,\vdots \cdots \vdots\, [T(\mathcal{B}_n)]_F \right]$

18. For any $v \in V$, we can choose any of the three bases $\mathcal{B}, \mathcal{S}, \mathcal{D}$.

For $v \in V$ rel to $\mathcal{B}$

$\mathrm{Iso}E^{-1}\, C\, \mathrm{Iso}B(v)$ — start here

$\mathrm{Iso}E^{-1}\, C\, \mathrm{Iso}B(\text{lc } \mathcal{B}\text{'s})$

$\mathrm{Iso}E^{-1}\, C\, \mathrm{Iso}B([[B_1 \cdots B_n]] \cdot [v]_B)$

$\mathrm{Iso}E^{-1}\, C\, \underbrace{\mathrm{Iso}B\,(v)}$

$\mathrm{Iso}E^{-1}\, \underbrace{C \cdot [v]_B}$

$\underbrace{\mathrm{Iso}E^{-1}\, [T(v)]_E}$

$[[E_1 \cdots E_n]] \cdot [T(v)]_E$

lc E's

$T(v)$

where $C = \mathrm{Ecol}^{-1}\, A\, \mathrm{Bcol} = T\mathrm{of}{}^{E}_{B}$

or $C = \mathrm{Ecol}^{-1}\, \mathrm{Fcol}\, G\, \mathrm{Dcol}^{-1}\, \mathrm{Bcol}$

or $C = P\, A\, P^{-1}$