M 212 Lect #2 1-25-10

The Why for SOV technique.

Suppose we have a one parameter family of solns of the form.

$$a(x) + b(y) = c$$

We diff & eliminate the parameter c

$$(a'(x) + 0)\,dx + 0 + b'(y)\,dy = 0$$

looks like

$$A(x)\,dx + B(y)\,dy = 0$$

$$\int A(x)\,dx + \int B(y)\,dy = C$$

Our Plan is to do 3 more methods

③ IF Integrating Factors

④ FOL First Order Linear

⑤ Bernoulli DE

all go back to exact.

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③ IF Integ Factors

Sometimes the $M\,dx + N\,dy = 0$ is not exact, but by tinkering with it, the DE becomes exact.

Ex $5 \sin y \, dx + x \cos y \, dy = 0$

$$\underset{M}{5\sin y}\,dx + \underset{N}{x\cos y}\,dy = 0$$

$$M_y = 5\cos y \neq N_x = \cos y$$

Suppose we mpy the DE by x^4

$$\underset{m}{5x^4 \sin y}\,dx + \underset{n}{x^5 \cos y}\,dy = 0$$

$$m_y = 5x^4 \cos y \equiv n_x = 5x^4 \cos y$$

Hence exact

The x^4 we just mpy'd by is called an IF integ factor for the given DE

So how might we find IF's like x^4

<u>The How:</u> For the DE $M\,dx + N\,dy = 0$

$\frac{M_y - N_x}{N}$ (and simplify) $\stackrel{\text{if}}{=}$ a fcn of x alone $= f(x)$

the IF will be $u = e^{\int f(x)dx} = e^{\int \frac{M_y - N_x}{N}dx}$

$\frac{N_x - M_y}{M}$ (and simplify) $\stackrel{\text{if}}{=}$ a fcn of y alone $= g(y)$

The IF will be $u = e^{\int g(y)dy} = e^{\int \frac{N_x - M_y}{M}dy}$

Return to our orig DE example

$$5\sin y \; dx + \underbrace{x\cos y}_{N} \, dy = 0$$

M

We already know $M_y \neq N_x$ so not exact

We seek an IF

$$\frac{M_y - N_x}{N} = \frac{5\cos y - \cos y}{x\cos y} = \frac{4\cos y}{x\cos y} = \frac{4}{x} = f(x)$$

$$\text{IF is } u = e^{\int \frac{4}{x}dx} = e^{4\int\frac{dx}{x}} = e^{4\ln x} = e^{\ln x^4}$$

$u = x^4$ is IF. We mpy DE by u

DE becomes $5x^4 \sin y \, dx + x^5 \cos y \, dy = 0$

We've already shown this to be exact on page 3 above.

Method (4) <u>First Order Linear</u> FOL

FOL looks like $A(x)y' + B(x)y = C(x)$

[This eq has an IF and can be solved that way.]

We seek a faster soln.

Divide all terms by $A(x)$

$$\frac{\cancel{A(x)}\,y'}{\cancel{A(x)}} + \frac{B(x)\,y}{A(x)} = \frac{C(x)}{A(x)}$$

$$\rightarrow \quad y' + P(x)\,y = Q(x)$$

The $\frac{M_y - N_x}{N}$ always work to give us this IF

$$\rightarrow \quad v = e^{\int P(x)\,dx}$$

Mpy DE by v

$$\underbrace{vy' + vP(x)\,y}_{\text{der of a prod}} = vQ(x) \quad \text{is exact}$$

$$v = e^{\int P(x)\,dx}$$

$$v' = e^{\int P(x)\,dx} \cdot P(x)$$

$$v\,P(x)$$

$$(vy)' = vQ(x)$$

$$\boxed{vy = \int vQ\,dx}$$

<u>Ex</u> of FOL

$$\frac{x^2 y'}{x^2} + \frac{3xy}{x^2} = \frac{18x^7}{x^2}; \quad y(1) = 5$$

$$\rightarrow \quad y' + \frac{3}{x} y = 18x^5$$

IF is $v = e^{\int +\frac{3}{x}dx} = e^{3\int \frac{dx}{x}} = e^{3\ln x}$

$$= e^{\ln x^3} = x^3$$

$vy = \int v Q \, dx$ means

$$x^3 y = \int x^3 \, 18x^5 \, dx$$

$$= \int 18x^8 dx = \cancel{18}^{2} \frac{x^9}{\cancel{9}} + c$$

$$x^3 y = 2x^9 + c$$

$$y = 2x^6 + c x^{-3}$$

$5 = 2 + c \quad$ so $c = 3 \qquad$ Sol to IVP

is $y = 2x^6 + 3x^{-3}$

5 Bernoulli DE

Bernoulli DE looks like FOL except the $Q(x)$ has a factor y^n

Ex

$$\frac{xy'}{x} + \frac{\frac{1}{x}y}{x} = \frac{x^2y^2}{x}$$

$$\frac{y'}{y^2} + \frac{\frac{1}{x^2}y}{y^2} = \frac{xy^2}{y^2}$$

$$y^{-2}y' + \frac{1}{x^2}\boxed{y^{-1}} = x$$

Let $z = \boxed{y^{-1}}$, $z' = (-1)y^{-2}\cdot y'$

$$(-1)y^{-2}y' + (-1)\frac{1}{x^2}y^{-1} = (-1)x$$

$$z' - \frac{1}{x^2}z = -x$$

FOL

Theory

$$y^{-n}y' + P(x)\,y\cdot y^{-n} = Q(x)\,y^n\,y^{-n}$$

$$y^{-n}y' + P(x)\,y^{1-n} = Q(x)$$

$z = y^{1-n}$ $\quad z' = (1-n)\,y^{-n}\cdot y'$

$$(1-n)y^{-n}y' + (1-n)P(x)y^{1-n} = (1-n)Q(x)$$

$$z' + (1-n)P(x)z = (1-n)Q(x)$$