M 212 Lect #7 2-12-10

Growth & Decay Problem

LNHCC or LNH — Undet. Coeff UC / Reduction of Order ROO / Variation of Params VOP

Differential Operators

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G & D Problem

Compounding Continuously

If $A$ = amount (\$)
$t$ = time (yrs)

The Amt grows at a rate proportional to the amount present at any time.

quoted interest rate

$$\frac{dA}{dt} = k \cdot A$$

$$\frac{dA}{dt} = r \cdot A \quad , \quad A(0) = A_0$$

Ex: $\boxed{\frac{dA}{dt} = .02A}$ , $A(0) = 3000$

$\frac{dA}{A} = .02\,dt$ Var are Sepd

$\int \frac{dA}{dt} = \int .02\,dt$

$\ln|A| = .02t + c$

$\ln A = .02t + c$

Apply IC

$\ln 3000 = .02(0) + c$

So soln is

$\ln A = .02t + \ln 3000$

$\ln A - \ln 3000 = .02t$

$\ln \frac{A}{3000} = .02t$

$e \qquad e$

$\frac{A}{3000} = e^{.02t}$

$\boxed{A = 3000e^{.02t}}$

$\boxed{t = \frac{\ln \frac{A}{3000}}{.02}}$

# Popular Questions for G + D Probs

1. How much ... A
   How much money is there at time # years
2. When will ... t
   When will the money double?
3. How fast ... $\frac{dA}{dt}$
   How fast when $t = 4$
4. Relatively How fast $\frac{A'}{A}$
   Rel how fast is the money growing at $t = 46.352$

| $t$ | $A$ | $\frac{dA}{dt}$ | $\frac{dA/dt}{A}$ |
|---|---|---|---|
| 4 | ___ | | |
| ___ | 6000 | | |
| 4 | ① ___ | ___ | |
| 46.352 | ① ___ | ② ___ | $\frac{②}{①}$ |

LNHCC or LNH

$y'' - 3y' - 4y = 0$ is LH (assoc LH)

$y'' - 3y' - 4y = 42e^{6x}$ LNH

1. Solve assoc LH

$m^2 - 3m - 4 = 0$

$(m-4)(m+1) = 0$

$m = 4, -1$

$y_c = c_1 e^{4x} + c_2 e^{-x}$

2. Now we do the Undetermined Coeffs UC Method

Let $y_t = ae^{6x}$ be a trial soln

$y_t' = 6ae^{6x}$

$y_t'' = 36ae^{6x}$

Plug into DE

$36ae^{6x} - 18ae^{6x} - 4ae^{6x} = 42e^{6x}$

$36a - 18a - 4a = 42$

$14a = 42$

$a = \frac{42}{14} = 3$

So we plug $a=3$ into $y_t$ to get

$y_p = 3e^{6x}$ (called a particular soln)

To get the general soln to LNH we add the $y_c$ and $y_p$

$y_g = y_c + y_p =$

$y_g = c_1 e^{4x} + c_2 e^{-x} + 3e^{6x}$

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We do the same LNH by ROO

Reduction of Order

$$y'' - 3y' - 4y = 42e^{4x}$$

Try to solve related LN

$$m^2 - 3m - 4 = 0$$

$$(m-4)(m+1) = 0$$

$$m = 4, -1$$

$$y_c = c_1 e^{4x} + c_2 e^{-x}$$

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ROO starts here.

$$\rightarrow \quad y_t = u(x) e^{4x}$$

$$y_t' = u \cdot e^{4x} \cdot 4 + u' e^{4x}$$

$$\rightarrow \quad y_t' = e^{4x}(4u + u')$$

$$y_t'' = e^{4x}(4u' + u'') + (4u + u') 4e^{4x}$$

$$\rightarrow \quad y_t'' = e^{4x}(u'' + 8u' + 16u)$$

Plug into LNH

$$e^{4x}(u'' + 8u' + 16u)$$
$$-3e^{4x}(\cdot u' + 4u)$$
$$-4e^{4x}(\quad u) = 42e^{6x}$$

$$e^{4x}(u'' + 8u' - 3u' + \underbrace{16u - 12u - 4u}_{\text{drop out}}) = 42e^{6x}$$
$$(u'' + 5u')$$

$$e^{4x}(u'' + 5u') = 42e^{6x}$$

Make a substitution $\begin{cases} z = u' \\ z' = u'' \end{cases}$

$$\frac{e^{4x}(z' + 5z)}{e^{4x}} = \frac{42e^{6x}}{e^{4x}}$$

The order has be reduced from 2 to 1

$$z' + 5z = 42e^{2x}$$

We can either repeat 2 page of ROO or use FOL from Chap 2

$z' + 5z = 42e^{2x}$ St form FOL p8

IF $v = e^{\int 5dx} = e^{5x}$

Usually $vy = \int vQ\,dx$

This time $vz = \int vQ\,dx$

$e^{5x} z = \int e^{5x} \cdot 42e^{2x}\,dx$

$= \int \frac{42}{7} e^{7x}\, 7\,dx$

$\frac{e^{5x} z}{e^{5x}} = \frac{6e^{7x} + c}{e^{5x}}$

$z = 6e^{2x} + c_1 e^{-5x}$ . But $z = u'$

$u' = 6e^{2x} + c_1 e^{-5x}$

$u = \int \left(\frac{6}{2} e^{2x} \cdot 2 + \frac{c_1}{-5} e^{-5x} \cdot (-5)\right) dx$

$u = 3e^{2x} + c_2 e^{-5x} + C_3$

$$u = 3e^{2x} + C_2 e^{-5x} + C_3$$

$$y_t = u \cdot e^{4x}$$

$$y_g = \left(3e^{2x} + C_2 e^{-5x} + C_3\right) e^{4x}$$

$$y_g = 3e^{6x} + C_2 e^{-x} + C_3 e^{4x}$$

is the general soln to LNH

# Differential Operators

$D^0 y = y = y(x)$

$Dy = y' = \frac{dy}{dx} =$

$D^2 y = y'' = \frac{d^2y}{dx^2} = DDy$

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So a second order DE (LNH)
that looks like

$$y'' - 3y' - 4y = 42e^{6x}$$

can be written

$$(D^2 - 3D - 4)\, y = 42e^{6x}$$

$$f(D)\, y = RHS = g(x)$$

Suppose we plug $y = e^{mx}$ into the DE

$$f(D)\,y = 0$$

$$f(c_1 e^{mx})$$

$$D^2 c_1 e^{mx} - 3D\,c_1 e^{mx} - 4c_1 e^{mx}$$

$$= c_1 m^2 e^{mx} - 3c_1 m e^{mx} - 4c_1 e^{mx}$$

$$= c_1 e^{mx}(m^2 - 3m - 4) \overset{\text{set}}{=} 0$$

So the only $m$'s that allow $c_1 e^{mx}$ to be a soln are the roots of the aux eq $m^2 - 3m - 4 = 0$

$f(m) = 0$ yield the expon of $e$ which is the soln