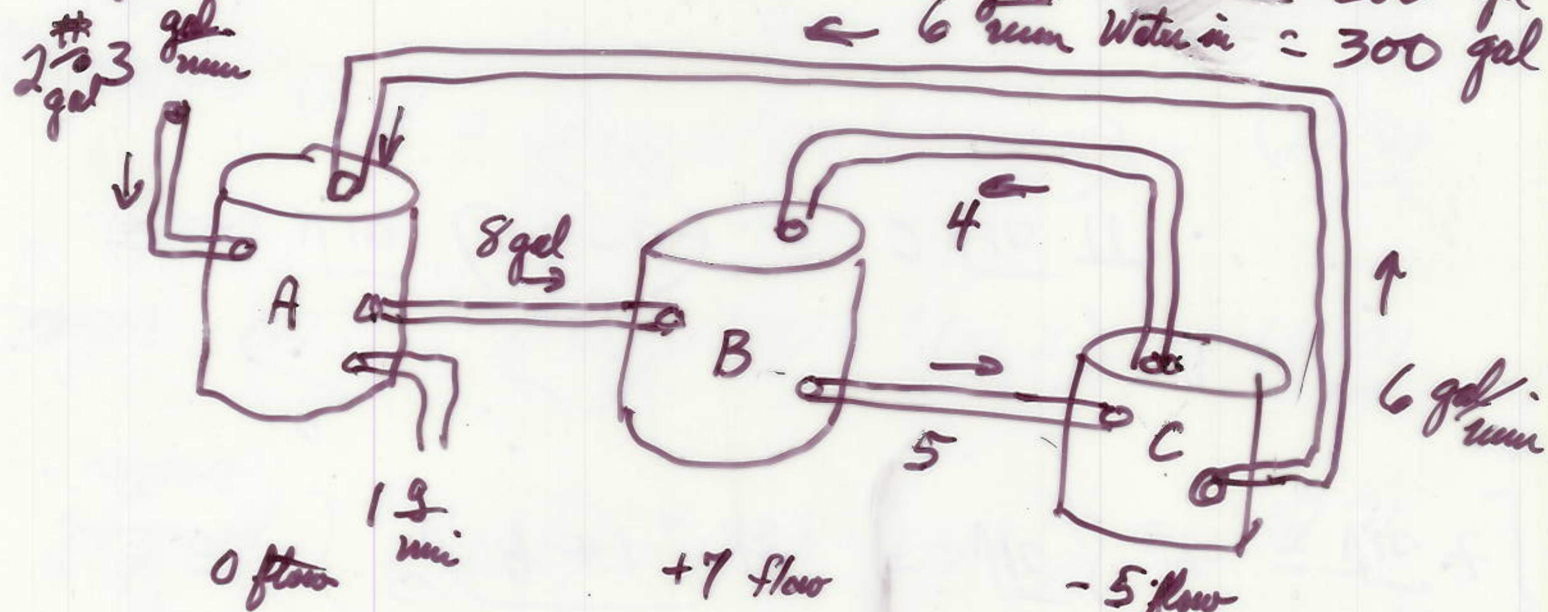
M212 Lect #10 3-1-10

System Mixture Problem

At time zero
Water in A = 100 gal
Water in B = 200 gal
Water in C = 300 gal



A, B, C are amts of salt in each tank @ any time t.

$$\frac{dA}{dt} = 2\cdot 3 + \frac{C}{300-5t}\cdot 6\,\tfrac{g}{m} - \frac{A}{100}\cdot 8 - \frac{A}{100}\cdot 1,$$

$$\frac{dB}{dt} = \frac{A}{100}\cdot 8 + \frac{C}{300-5t}4 - \frac{B}{200+7t}5,$$

$$\frac{dC}{dt} = \frac{B}{200+7t}5 - \frac{C}{300-5t}4 - \frac{C}{300-5t}6,$$

$A(0) = 20 \quad B(0) = 0 \quad C(0) = 30$

Diagram for CEDE

$$
\begin{tikzcd}[column sep=huge, row sep=huge]
\text{(1) CEDE in } x\text{'s} \arrow[r, "y = x^m"] \arrow[d, "\begin{array}{l} \left.\begin{array}{l} x = e^t \\ t = \ln x \end{array}\right\} \text{for } x>0 \\ \left.\begin{array}{l} x = -e^t \\ t = \ln(-x) \end{array}\right\} \text{for } x<0 \end{array}"] & \text{(2) get } m\text{'s} \arrow[r, "\text{if } m\text{'s are}", "\text{real \& distinct}"'] \arrow[d] & \text{(5) Write } \cancel{\text{G Soln}} \text{ as a LC of solns} \\
\text{LHCC in } t\text{'s} \arrow[r, "\text{solve aux}", "\text{eqn}"'] & \text{get } m\text{'s (3)} \arrow[r, "\text{write}", "\text{gen soln}"'] & \text{gen'l soln (4)} \arrow[u]
\end{tikzcd}
$$

Suppose we are at (2) and just got

(2) $m = 5, 5, 5$

(3) $m = 5, 5, 5$

$(e^t)^5$

(4) $y_c = c_1 e^{5t} + c_2 t e^{5t} + c_3 t^2 e^{5t}$

$= (c_1 + c_2 t + c_3 t^2) e^{5t}$

(5) $y_c = \left(c_1 + c_2 \ln x + c_3 (\ln x)^2\right) x^5$

A complex example

At (2) $m = 5 \pm 3i,\ 5 \mp 3i$

(3) $m = 5 \pm 3i,\ 5 \pm 3i$

(4) $y_c = e^{5t}\Big(c_1 \cos(3t) + c_2 \sin(3t)$

$+ c_3\, t \cdot \cos(3t) + c_4\, t \sin(3t) \Big)$

(5) Convert using $x = e^t \quad t = \ln x$

$y_c = x^5 \Big[c_1 \cos(3 \ln x) + c_2 \sin(3 \ln x)$

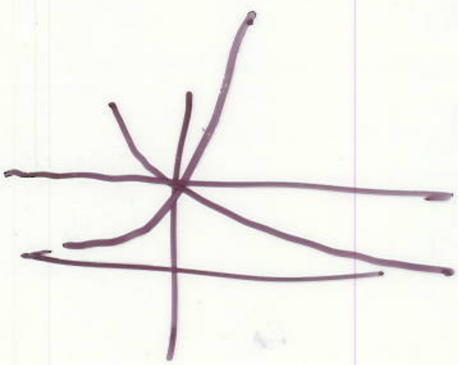
$+ c_3 \ln x \cdot \cos(3 \ln x) + c_4 \ln x \cdot \sin(3 \ln x) \Big]$

Some LI sets of fcns

$\{1, x, x^2\}$ LI

$$c_1 1 + c_2 x + c_3 x^2 = 0$$

$\{e^{2x}, e^{-x}, e^{0}\}$ LI



$\{\cos x, \sin x, 1\}$ LI

Can we det whether

$\{x^2+1,\ x^2-3x,\ 2x^2-3x+1\}$ are LD or LI ?

$$a(x^2+1) + b(x^2-3x) + c(2x^2-3x+1) \equiv 0$$

$$(a+b+2c)x^2 + (-3b-3c)x + (a+c\ )\cdot 1 \equiv 0$$

$$\begin{cases} a+b+2c=0 \\ -3b-3c=0 \\ a \quad\ \ + c=0 \end{cases} \rightarrow \quad \begin{aligned} b&=-c \\ a&=-c \end{aligned}$$

Let $c=1$

$b=-1$

$a=-1$

Check it in eq 1 $\quad -1-1+2(1)=0$

Show it

$$-1(x^2+1) - 1(x^2-3x) + 1(2x^2-3x+1) \equiv 0$$

This is the long way "using the definition" to conclude that the three fcns are LD

Use Th. 4.3 to det LD or LI

$\{x^2+1,\ x^2-3x,\ 2x^2-3x+1\}$

LH?? $m = 0, 0, 0.$ $\quad e^{0x} = 1$

$m^3 = 0$

$D^3 y = 0$ $\quad C_1 1 + C_2 x \cdot 1 + C_3 x^2 \cdot 1$

$y''' = 0$ has these three fcns as solns

$$W(x^2+1,\ x^2-3x,\ 2x^2-3x+1)$$

$$= \begin{vmatrix} x^2+1 & x^2-3x & 2x^2-3x+1 \\ 2x & 2x-3 & 4x-3 \\ 2 & 2 & 4 \end{vmatrix}$$

$$= +(x^2+1)\begin{vmatrix} 2x-3 & 4x-3 \\ 2 & 4 \end{vmatrix} - (x^2-3x)\begin{vmatrix} 2x & 4x-3 \\ 2 & 4 \end{vmatrix} + (2x^2-3x+1)\begin{vmatrix} 2x & 2x-3 \\ 2 & 2 \end{vmatrix}$$

$$= +(x^2+1)\left[(2x-3)\cdot 4 - 2(4x-3)\right] - (x^2-3x)\left[2x\cdot 4 - 2(4x-3)\right]$$

$$+ (2x^2-3x+1)\left[2\cdot 2x - 2(2x-3)\right] \equiv 0$$

So LD

Outline of Theory of Linear DE's

		Def IVP
LH or LNH	4.1	LNHContC,LCNZ IVP has a unique solution. * PS
		Def BVP, LC
LH	4.2	LC of solns is soln (Superposition Principle for LH).
		Def trivial LC, LD, LI, Wronskian
LH	4.3	For n solns to n th order LH, W not 0 <==> LI. ← PS
LH		Def: Fundamental Set of Solutions < LI, n of them
LH	4.4	There exists a fundamental set of solns for LH
LH	4.5	General Solution to LH is LC of fundamental set $c_1 e^x + c_2 e^{5x}$
LNH	4.6	$y_g = y_c + y_p$ ✓
LNH	4.7	Superposition principle for LNH (y_p's and g(x)'s)

$$y'' - 3y' - 4y = e^{2x} + \sin(x)$$