M 212 Lect #19 4-7-10

Convolution Theorem

$$\mathcal{L}\left\{\int_0^t f(t-\beta)\,g(\beta)\,d\beta\right\} = F(s)\cdot G(s)$$

$f(t)$	$F(s)$
$f(t) * g(t) \leftarrow$	$F(s)\cdot G(s)$

$$\downarrow \qquad \downarrow$$

$$f(t-\beta) \qquad g(\beta)$$

or

$$\downarrow \qquad \downarrow$$

$$f(\beta) \qquad g(t-\beta)$$

$$= \int_0^t f(t-\beta)\,g(\beta)\,d\beta$$

Find

$$\mathcal{L}^{-1}\left\{ \frac{1}{s-2} \cdot \frac{1}{s+5} \right\} = \mathcal{L}^{-1}\left\{ \frac{1}{s^2+3s-10} \right\}$$

P2

$=$

$=$

$=$

	$F(s)$
$e^{2t} * e^{-5t} \leftarrow$	$\frac{1}{s^2+3s-10} = \frac{1}{(s-2)(s+5)} = \frac{1/7}{s-2} + \frac{1/-7}{s+5}$ $2 \qquad -5$ $\frac{1}{s-2} \cdot \frac{1}{s+5}$

$$= \int_0^t e^{2(t-\beta)} e^{-5\beta}\, d\beta$$

$$= \int_0^t$$

$u = -7\beta$
$du = -7\,d\beta$

$$= \frac{e^{2t}}{-7} \int_0^t e^{-7\beta} (-7)\, d\beta$$

$$= \frac{e^{2t}}{-7} \left[e^{-7\beta} \right]_0^t = \frac{e^{2t}}{-7} \left[e^{-7t} - 1 \right]$$

$$= -\frac{1}{7} e^{-5t} + \frac{1}{7} e^{2t}$$

e^{0t} or t^0

$e^{0(t-\beta)}$ | $F(s)$

$1 * f(t)$ $\leftarrow$ $\frac{1}{s}F(s)$

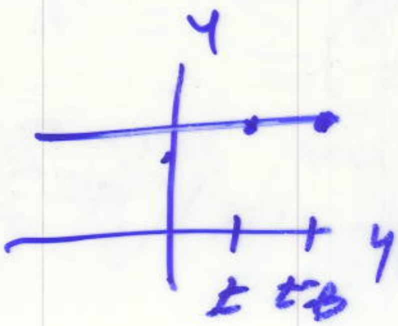
$$= \int_0^t 1 \cdot f(\beta)\, d\beta$$

$$= \int_0^t f(\beta)\, d\beta$$

$\int_0^t \sin(3\beta)\, d\beta$ $\rightarrow$ $\frac{1}{s} \cdot \frac{3}{s^2+9}$

$$\int_0^t 1 \cdot \sin(3\beta)\, d\beta$$

$1 * \sin(3t)$



Proof of Der of Lap Th

$$\mathcal{L}\{t^n f(t)\} \stackrel{?}{=} (-1)^n \frac{d^n}{ds^n} F(s)$$

$$= \int_0^\infty e^{-st} t^n f(t)\, dt$$

$$= (-1)^n \frac{d^n}{ds^n} \mathcal{L}\{f(t)\}$$

$$= (-1)^n \frac{d^n}{ds^n} \int_0^\infty e^{-st} f(t)\, dt$$

$$= \int_0^\infty (-1)^n \underbrace{\frac{d^n}{ds^n} e^{-st}} f(t)\, dt$$

$$= \int_0^\infty (-1)^n (-t)^n e^{-st} f(t)\, dt$$

$$= \int_0^\infty e^{-st} t^n f(t)\, dt$$

<u>aside</u>

$$\frac{d}{ds} e^{-st} = e^{-st}(-t)$$

$$\frac{d^2}{ds^2} e^{-st} = e^{-st}(-t)(-t)$$

$$= e^{-st}(-t)(-t)(-t)$$

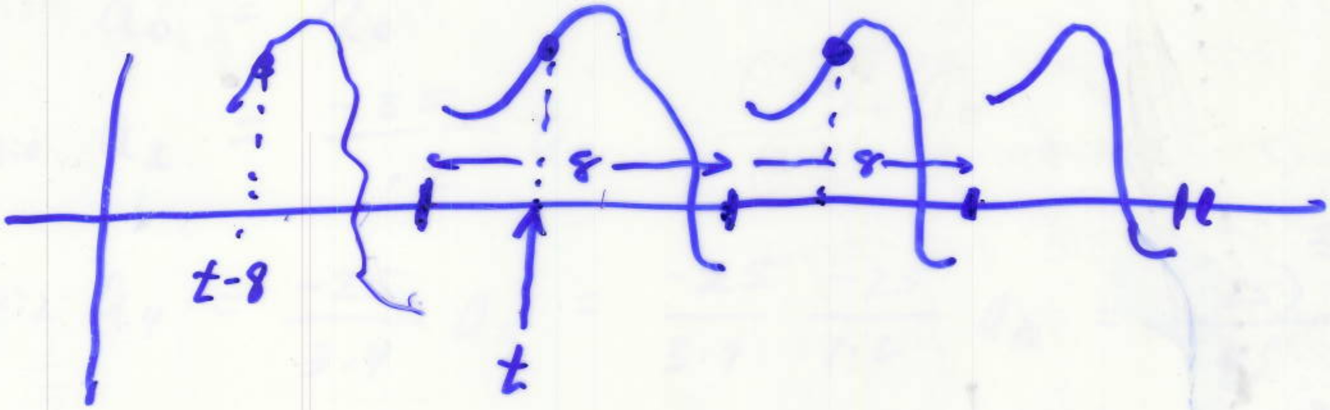
$$\vdots$$

$$\frac{d^n}{ds^n} e^{-st} = e^{-st}(-t)^n$$

$$\equiv$$

Periodic Function Th

Let f be periodic with period ω.



$$f(t \pm 8) = f(t) \qquad f(t) = f(t+8)$$

or in general

$$f(t+\omega) = f(t)$$

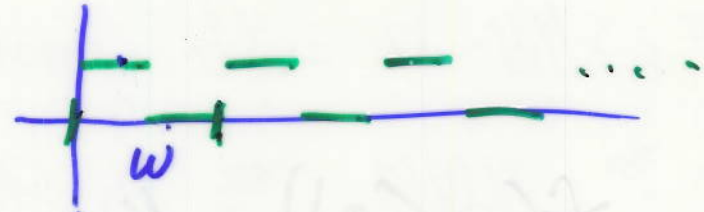
Some interesting per fcns are

$\sin t$ 2π 

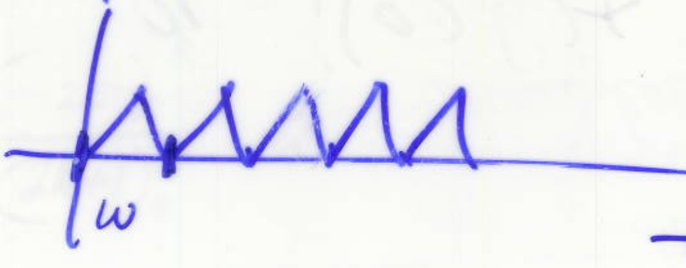
$\cos t$ 2π 

$\tan t$ π 

square wave



triangle wave



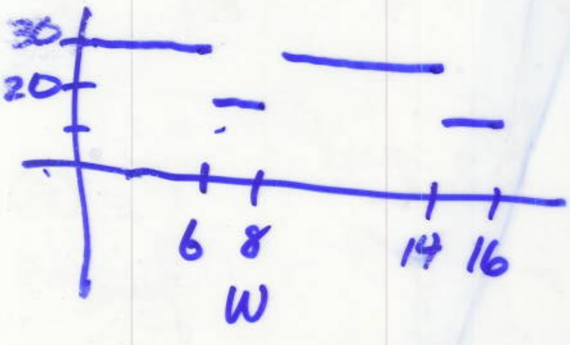
$\sin^2$. 

Per Fcn Th

$$\mathcal{L}\{f(t)\} = \frac{1}{1-e^{-sw}} \int_0^w e^{-st} f(t)\,dt$$

when $f(t) = f(t+w)$

$f(t)$	$F(s)$
$f(t) = f(t+8)$	
$f(t) = \begin{cases} 30 & 0 \le t < 6 \\ 20 & 6 \le t < 8 \end{cases}$	$\Rightarrow \dfrac{1}{1-e^{-8s}} \displaystyle\int_0^8 e^{-st} f(t)\,dt$



$$= \frac{1}{-s}\frac{1}{1-e^{-8s}}\left[\int_0^6 30e^{-st}(-s)\,dt + \int_6^8 20e^{-st}(-s)\,dt\right]$$

$$= \frac{1}{-s}\frac{1}{1-e^{-8s}}\left\{\Big[30e^{-st}\Big]_0^6 + \Big[20e^{-st}\Big]_6^8\right\}$$

$$= \frac{1}{-s}\frac{1}{1-e^{-8s}}\left\{30\left(e^{-6s}-1\right) + 20\left(e^{-8s}-e^{-6s}\right)\right\}$$

and a little more simplification

Systems of IVP's by $\mathcal{L}T$

$$\begin{cases} x' = 3x - y - 12 & x(0) = 0 \\ y' = x + y + 4e^t & y(0) = 1 \end{cases}$$

Let $X(s) = \mathcal{L}\{x(t)\}$ Find $x(t)$

$Y(s) = \mathcal{L}\{y(t)\}$

$$1\left(sX - \cancel{x(0)}^{\,0}\right) = 3X - Y - \frac{12}{s}$$

$$1.\left(sY - \cancel{y(0)}^{\,1}\right) = X + Y + \frac{4}{s-1}$$

$$\left\{\begin{aligned} (s-3)X + Y &= -\frac{12}{s} \\ -X + (s-1)Y &= \frac{(s-1)}{(s-1)}1 + \frac{4}{s-1} \end{aligned}\right\} \text{ good form}$$

$$= \frac{s+3}{s-1}$$

$$X = \frac{\begin{vmatrix} -\frac{12}{s} & 1 \\ \frac{s+3}{s-1} & s-1 \end{vmatrix}}{\begin{vmatrix} s-3 & 1 \\ -1 & s-1 \end{vmatrix}}$$

$$X = \frac{\frac{(s-1)(-12)}{(s-1)s}(s-1) - \frac{s+3}{s-1}\frac{s}{s}}{(s-3)(s-1) - (-1)(1)}$$

$$= \frac{\frac{-12(s^2-2s+1) - (s^2+3s)}{(s-1)s}}{s^2-4s+4} = \frac{?}{(s-2)^2}$$

$$= \frac{-13s^2+21s-12}{s(s-1)(s-2)^2}$$