M 212 Lect #20 4-12-10

Fcn that look like this always have $\mathcal{L}T$'s

not grow too fast

t_0

want $f(t)$ to be piecewise cont on all finite intervals $[0, t_0]$

want $f(t)$ to be (cont) and of exponential order α on (t_0, ∞)

$e^{Mt} \geq |f(t)|$ for $t >$ some t_0

$$f(t) = \begin{cases} - & 0 \le t < 1 \\ - & 1 \le t < 5 \\ - & \end{cases}$$

Idea

$$\int_0^\infty e^{-st} f(t)\, dt$$

$$= \int_0^{t_0} \qquad \int_{t_0}^{\infty} e^{-st} f(t)\, dt$$

I e^{30t}

II e^{t^2}

Proof of Lap of Der Th

$\mathcal{L}\{f'(t)\}$ (y') $\stackrel{?}{=} s\,F(s) - f(0)$ ($s\,Y(s) - y(0)$)

$$= \int_0^\infty e^{-st} f'(t)\, dt$$

$u = e^{-st}$ $\quad dv = f'(t)\,dt$

$du = e^{-st}(-s)\,dt$ $\quad v = f(t)$

$$= uv - \int v\,du$$

$$= e^{-st} f(t)\Big]_0^\infty - \int_0^\infty e^{-st}(-s)\, f(t)\, dt$$

$$= \int_0^\infty e^{-st}(s) f(t)\,dt + \lim_{B\to\infty} e^{-sB} f(t)\Big]_0^B$$

$$= \int_0^\infty e^{-st}(s)\, f(t)\,dt + \lim_{B\to\infty} e^{-sB} f(B) - e^{-s(0)} f(0)$$

$e^{-sB}f(B) \to 0$ because f is assumed to be of exp order.

and $s > 0$

$e^{-s(0)}f(0) = f(0)$

$$\equiv \; = s\int_0^\infty e^{-st} f(t)\,dt - f(0)$$

$$= \int_0^\infty s\, e^{-st} f(t)\,dt - f(0)$$

Proof of Convolution Th

$$\mathcal{L}\left\{\int_0^t f(t-\beta)\,g(\beta)\,d\beta\right\} \overset{?}{=} F(s)\cdot G(s)$$

$$= \int_0^\infty e^{-st}\int_0^t f(t-\beta)\,g(\beta)\,d\beta\,dt$$

$$= \int_0^\infty\int_0^t e^{-st} f(t-\beta)\,g(\beta)\,d\beta\,dt$$

$$= \int_0^\infty e^{-s\alpha} f(\alpha)\,d\alpha \cdot \int_0^\infty e^{-s\beta} g(\beta)\,d\beta$$

$$= \int_0^\infty\int_0^\infty e^{-s(\alpha+\beta)} f(\alpha)\,g(\beta)\,d\alpha\,d\beta$$

$t = \alpha+\beta$

$\alpha = t-\beta$ ←

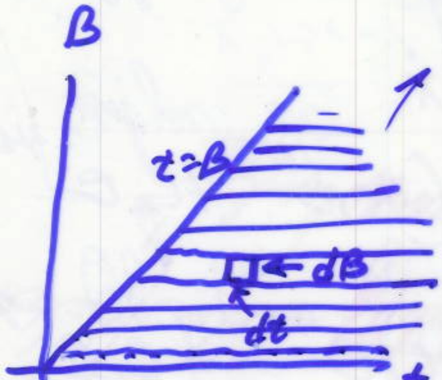
$d\alpha = dt$

$\alpha = 0 \Rightarrow t = \beta$

$\alpha \to \infty \Rightarrow t \to \infty$

$$= \int_0^\infty\int_\beta^\infty e^{-st} f(t-\beta)\,g(\beta)\,dt\,d\beta$$

$$= \int_0^\infty\int_0^t e^{-st} f(t-\beta)\,g(\beta)\,d\beta\,dt$$



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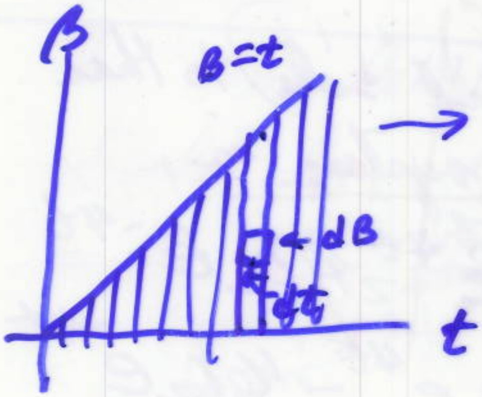
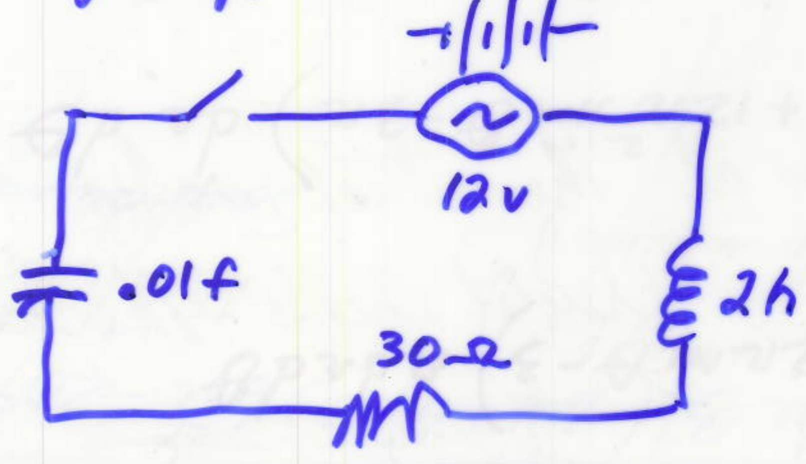


Integrodifferential Equation



$$\boxed{2q'' + 30q' + 100q = 12 \,, \quad \overset{q_0}{q(0) = .4\,c} \quad q'(0) = 0}$$

2nd order IVP, dep var is q

$i(0) = 0$

Since q' is current, $q' = i$

$$\frac{dq}{dt} = i \qquad \int_{.4}^{q} dq = \int_0^t i(t)\,dt$$

$$q\Big]_{.4}^{q(t)} = \int_0^t i(t)\,dt$$

$$q(t) - \cancelto{q_0}{.4} = \int_0^t i(t)\,dt$$

$$q(t) = q(t) = q_0 + \int_0^t i(t)\,dt$$

So our IVP becomes

$$2i' + 30i + 100\left(q_0 + \int_0^t i(\beta)\,d\beta\right) = 12,$$

$$i(0) = i_0 = 0$$

an example of an integrodifferential equation.

Taking the $\mathcal{L}T$ we get

$$2(sI - i_0) + 30I + \frac{100q_0}{s} + 100\left(\frac{1}{s}I\right) = \frac{12}{s}$$