M 212 Lect #26 5-3-10

Recall how we got Taylor Polynomials for a given fcn $f(x)$ and a value x_0

We want to create a third degree polynomial that matches the <u>ht</u>, <u>slope</u>, <u>mood</u> & <u>moodswing</u> of the fcn at x_0

$$f(x) \approx p_3(x) = a_0 + a_1(x-x_0) + a_2(x-x_0)^2 + a_3(x-x_0)^3$$

We insist that

$$f(x_0) = p_3(x_0) = a_0 + a_1(x_0-x_0) + a_2(x_0-x_0)^2 + a_3(x_0-x_0)^3$$

$$\boxed{\text{So } a_0 = f(x_0) = \frac{f(x_0)}{0!}}$$

$$p_3'(x) = 0 + a_1 + 2a_2(x-x_0) + 3a_3(x-x_0)^2$$

We insist that

$$f'(x) = p_3'(x_0) = 0 + a_1 + 0 + 0$$

$$\boxed{\text{So } a_1 = \frac{f'(x_0)}{1} = \frac{f'(x_0)}{1!}}$$

$$P_3''(x) = 0 + 0 + 2a_2 \quad + 2\cdot 3\, a_3 (x - x_0)$$

We insist that

$$f''(x_0) = P_3''(x_0) = 2a_2$$

$$\boxed{\text{So } a_2 = \frac{f''(x_0)}{1\cdot 2} = \frac{f''(x_0)}{2!}}$$

Similarly

$$\boxed{\text{So } a_3 = \frac{f'''(x_0)}{1\cdot 2\cdot 3} = \frac{f'''(x_0)}{3!}}$$

So the best poly approx of $f(x)$ near x_0 of degree 3 is this $P_3(x)$

$$f(x) \approx P_3(x) = \frac{f(x_0)}{0!} + \frac{f'(x_0)}{1!}(x - x_0) + \frac{f''(x_0)}{2!}(x - x_0)^2 + \frac{f'''(x_0)}{3!}(x - x_0)^3$$

Similarly the best n^{th} order poly for $f(x)$ is $f(x) \approx p_n(x) = \sum_{k=0}^{n} \frac{f^{(k)}(x_0)}{k!}(x-x_0)^k$

or the Taylor Series

$$f(x) = \sum_{k=0}^{\infty} \frac{f^{(k)}(x_0)}{k!}(x-x_0)^k$$

Taylor ~~Series~~ is unsurpassed in approximating a fcn $f(x)$ near x_0. But it is generally lousy in approx $f(x)$ on intervals (containing x_0.)

This is where Fourier Series comes in.

To do Fourier Series we start with a fcn $f(x)$ and write the general F.S. with coeff not yet determined

$$f(x) \approx \mathcal{F}_3(x) = \frac{a_0}{2} + a_1 \cos\frac{1\pi x}{p} + a_2 \cos\frac{2\pi x}{p}$$
$$+ a_3 \cos\frac{3\pi x}{p}$$
$$+ b_1 \sin\frac{1\pi x}{p} + b_2 \sin\frac{2\pi x}{p}$$
$$+ b_3 \sin\frac{3\pi x}{p}$$

We insist that

$$\int_{-p}^{p} 1\cdot f(x)\,dx = \int_{-p}^{p} 1\cdot \mathcal{F}_3(x)\,dx$$

$$= \frac{a_0}{2}\int_{-p}^{p} 1\cdot 1\,dx + a_1\int_{-p}^{p} 1 \cos\frac{1\pi x}{p}dx + a_2\int_{-p}^{p} 1\cdot\cos\frac{2\pi x}{p}\,dx$$

$$+ a_3\int_{-p}^{p} 1\cdot\cos\frac{3\pi x}{p}dx +$$ (next page)

$$= \text{prev page} + b_1 \int_{-p}^{p} 1 \sin\frac{1\pi x}{p}\,dx + b_2 \int_{-p}^{p} 1\cdot \sin\frac{2\pi x}{p}\,dx$$

$$+ \, b_3 \int_{-p}^{p} 1 \sin\frac{3\pi x}{p}\,dx$$

$$\int_{-p}^{p} 1\cdot f(x)\,dx = \frac{a_0}{2}\cancelto{2p}{\langle 1, 1\rangle} + a_1 \cancelto{0}{\left\langle 1, \cos\frac{1\pi x}{p}\right\rangle} + \cancelto{0}{5 \text{ more terms}}$$

$$= \frac{a_0}{\cancel{2}}\cdot \cancel{2}p \qquad \text{So}$$

$$\boxed{a_0 = \frac{1}{p}\int_{-p}^{p} 1\cdot f(x)\,dx}$$

We insist that

$$\int_{-p}^{p} \cos\frac{1\pi x}{p} f(x)\,dx = \int_{-p}^{p} \cos\frac{1\pi x}{p}\, \mathcal{F}_3(x)\,dx$$

$$= 0 + a_1 \cancelto{p}{\left\langle \cos\frac{1\pi x}{p}, \cos\frac{1\pi x}{p}\right\rangle} + 5 \text{ more zero terms}$$

$$= a_1 p$$

So $a_1 = \frac{1}{p} \int_{-p}^{p} \cos \frac{\pi x}{p} f(x)\, dx$

Similarly $\boxed{a_n = \frac{1}{p} \int_{-p}^{p} \cos \frac{n\pi x}{p} f(x)\, dx}$

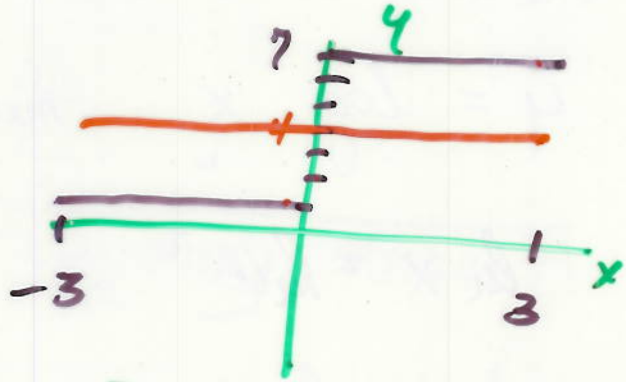
and

$$\boxed{b_n = \frac{1}{p} \int_{-p}^{p} \sin \frac{n\pi x}{p} f(x)\, dx}$$

Example of Fourier Series

Find the F.S. for this f(x) on the interval

$$f(x) = \begin{cases} 1 & -3 \le x < 0 \\ 7 & 0 \le x < 3 \end{cases}$$



$$a_0 = \frac{1}{p} \int_{-p}^{p} 1 \cdot f(x)\,dx = \frac{1}{3} \int_{-3}^{3} 1 \cdot f(x)\,dx$$

$$= \frac{1}{3}\left[\int_{-3}^{0} 1 \cdot 1\,dx + \int_{0}^{3} 1 \cdot 7\,dx \right] =$$

$$= \frac{1}{3}\left[3 + 21 \right] = 8$$

$$F_0(x) = \frac{a_0}{2} = \frac{8}{2} = 4$$

$$a_1 = \frac{1}{p}\int_{-p}^{p} \cos\frac{1\pi x}{p} f(x)\,dx$$

$u = \frac{\pi x}{3}$
$du = \frac{\pi}{3}dx$

$$= \frac{3}{\pi}\frac{1}{3}\left[\int_{-3}^{0} \cos\frac{1\pi x}{3}\cdot 1\,\frac{\pi}{3}dx + \int_{0}^{3} \cos\frac{1\pi x}{3}\cdot 7\,\frac{\pi}{3}dx\right]$$

$$= \frac{1}{\pi}\left[\left[\sin\frac{\pi x}{3}\right]_{-3}^{0} + 7\sin\frac{\pi x}{3}\Big]_{0}^{3}\right]$$

$$= \frac{1}{\pi}\left\{\cancel{\sin(0)}^{\,0} - \cancel{\sin(-\pi)}^{\,0} + 7\left[\cancel{\sin\pi}^{\,0} - \cancel{\sin 0}^{\,0}\right]\right\} = \frac{0}{\pi} = 0$$

$$b_1 = \frac{1}{p}\int_{-p}^{p} \sin\frac{1\pi x}{p} f(x)\,dx$$

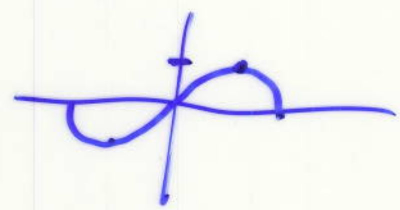
$$= \frac{3}{\pi}\frac{1}{3}\left[\int_{-3}^{0}\sin\frac{\pi x}{3}\cdot 1\,\frac{\pi}{3}dx + 7\int_{0}^{3}\sin\frac{\pi x}{3}\,\frac{\pi}{3}dx\right]$$

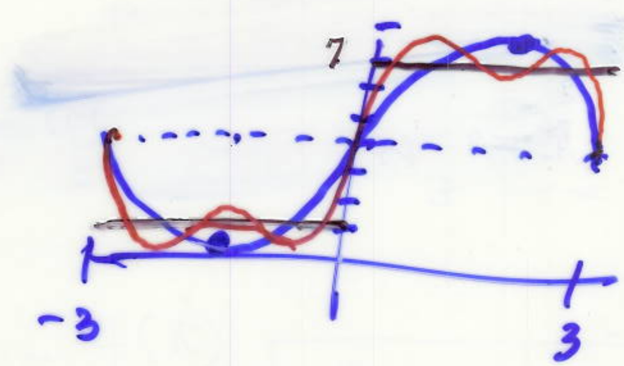
$$= -\frac{1}{\pi}\left[\left[\cos\frac{\pi x}{3}\right]_{-3}^{0} + 7\cos\frac{\pi x}{3}\Big]_{0}^{3}\right]$$

$$= -\frac{1}{\pi}\left[1-(-1) + 7(-1-1)\right] = \frac{12}{\pi} \approx 3.8$$

$$\mathcal{F}_1(x) = \frac{a_0}{2} + a_1 \cos\frac{1\pi x}{3} + b_1 \sin\frac{1\pi x}{3}$$

$$\doteq 4 + 0 \cos\frac{\pi x}{3} + 3.8 \sin\frac{\pi x}{3}$$





$$\mathcal{F}_1(x) \doteq 4 + 3.8 \sin\frac{1\pi x}{3} + 1.1 \sin\frac{2\pi x}{3}$$