M 212 Lect #28 5-10-10

The soln of the heat eqn.

$$kU_{xx} = U_t \qquad U(x,0) = f(x)$$

$$U(0,t) = 0$$

$$U(L,t) = 0$$

Let $U = XT$ be a trial prod soln

$$U_x = X'T \qquad U_{xx} = X''T \qquad U_t = XT'$$

$$\frac{k\,X''T}{XT} = \frac{XT'}{XT}$$

$$\frac{\cancel{k}\,X''}{X} = \frac{T'}{kT}$$ vars are sep'd

$$\underbrace{k\,\frac{X''}{X}} = K = \underbrace{\frac{T'}{T}}$$

$$\frac{X''}{X} = K$$

$$X'' = KX$$

$$X'' - KX = 0$$

$$K = \frac{T'}{kT}$$

$$kK = \frac{dT}{dt\,T}$$

$$T = C_5 e^{kKt}$$

Since this DE depends on K pos, zero or neg

Case I K pos write $K = \lambda^2$

Case II K zero write $K = 0$

Case III K neg write $K = -\lambda^2$

Case I. $K = \lambda^2$

$$X'' - \lambda^2 X = 0$$

$$m^2 - \lambda^2 = 0 \implies m = \pm\lambda$$

$$X = c_3 e^{\lambda x} + c_4 e^{-\lambda x}$$

$$\text{and } T = c_5 e^{k\lambda^2 t}$$

So Gen Prod Soln for Case I is

$$u(x,t) = e^{k\lambda^2 t}\left(c_1 e^{\lambda x} + c_2 e^{-\lambda x}\right)$$

Apply BC's $u(0,t)$

$$e^{k\lambda^2 t}(c_1 + c_2) = 0$$

$c_1 + c_2 = 0$ $\qquad c_2 = -c_1$

$$u(L,t) = 0$$

$$e^{k\lambda^2 t}\left(c_1 e^{\lambda L} - c_1 e^{-\lambda L}\right) = 0$$

$$c_1 e^{k\lambda^2 t}\left(e^{\lambda L} - e^{-\lambda L}\right) = 0$$

$C_1 = 0$ $\quad$ $\underline{\underline{or}}$ $\quad$ $e^{\lambda L} - e^{-\lambda L} = 0$

$\downarrow$ $\qquad\qquad\qquad\qquad \vdots$

$C_2 = -C_1 = 0$ $\qquad\qquad \lambda L = 0$

So

$u(x,t) = 0$ $\qquad$ $\lambda = 0$ $\qquad$ $L = 0$

But $\lambda \neq 0$ by assump. $\qquad$ $L \neq 0$ by hyp.

So $u(x,t) \equiv 0$

Apply other BC $\quad u(x,0) = f(x)$

This cond can only be met if no candles heated the rod.

So Case I yields nothing.

Case II $\quad K = 0 \quad \lambda = 0$

$X'' - 0X = 0 \qquad X'' = 0$

$m^2 - 0 = 0 \qquad m = 0, 0$

$X = c_1 + c_2 x$

$u(x,t) = XT = (c_1 + c_2 x)\, e^0$

Apply BC's $\quad u(0,t) = 0$

$c_1 + c_2 \cdot 0 = 0 \Rightarrow c_1 = 0$

$u(L,t) = 0$

$c_2 L = 0$

$c_2 = 0$ or $L = 0$ #

$u(x,t) = 0$. Similarly to Case I

Case 2 yields nothing.

Case III $\quad K = -\lambda^2$

$$T = c_5 e^{-k\lambda^2 t}$$

$$X'' + \lambda^2 X = 0$$

$$m^2 + \lambda^2 = 0$$

$$m = \pm \lambda i$$

$$X = c_3 \cos \lambda x + c_4 \sin \lambda x$$

So $\quad u(x,t) = XT$

$$u(x,t) = (c_1 \cos \lambda x + c_2 \sin \lambda x) e^{-k\lambda^2 t}$$

Apply BC $\quad u(0,t) = 0$

$$(c_1 \cdot 1 + c_2 \cdot 0) e^{-k\lambda^2 t} = 0$$

So $c_1 = 0$

$$u(x,t) \overset{\text{now}}{=} c_2 \sin \lambda x \, e^{-k\lambda^2 t}$$

$$u(L,t) = 0$$

$$C_2 \sin \lambda L \, e^{-k\lambda^2 t} = 0$$

$$C_2 \sin \lambda L = 0$$

$C_2 = 0$ or $\sin(\lambda L) = 0$

$$\lambda L = n\pi \qquad n = 1, 2, 3 \ldots$$

$u(x,t) \equiv 0$

$$\lambda = \frac{n\pi}{L}$$

So soln must be

$$u(x,t) = X T$$

$$u_n(x,t) = C_2 \sin \frac{n\pi x}{L} \, e^{-k\left(\frac{n\pi}{L}\right)^2 t}$$

BC3 is $u(x,0) = f(x)$

$$c_2 \sin \frac{n\pi x}{L} = f(x)$$

Use Superpos. prin. to create

$$u(x,t) = \sum_{n=1}^{\infty} u_n(x,t)$$

$$= \sum_{n=1}^{\infty} A_n \sin \frac{n\pi x}{L} e^{-k\left(\frac{n\pi}{L}\right)^2 t}$$

Apply $u(x,0) = f(x)$

$$f(x) = \sum_{n=1}^{\infty} A_n \sin \frac{n\pi x}{L}$$

But $f(x)$ has a Fourier Series

$$f(x) = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{L}$$

So the A_n's that will work will be the Fourier sin coeffs

$$b_n = \frac{1}{p}\int_{-p}^{p} \sin\frac{n\pi x}{p} f(x)\,dx$$

$$A_n = \frac{2}{L}\int_{0}^{L} \sin\frac{n\pi x}{L} f(x)\,dx$$

So the soln to the heat eqn is

$$\boxed{u(x,t) = \sum_{n=1}^{\infty} \frac{2}{L}\int_{0}^{L} \sin\frac{n\pi x}{L} f(x)\,dx \cdot \sin\frac{n\pi x}{L}\, e^{-k\left(\frac{n\pi}{L}\right)^2 t}}$$

Final Exam — here

Monday May 17 5:00 – 7:30

1. Fill out the card — written by hand

2. Re-work all HQ's using card

3. Re-view all PS's

4. Eat & Sleep

5. For your last hour of prep,
study only what you know.