Graphs and Euler Circuits

Graph Theory and Network Flows

In the modern world, planning efficient routes is essential for business and industry, with applications as varied as product distribution, laying new fiber optic lines for broadband internet, and suggesting new friends within social network websites like Facebook.

This field of mathematics started nearly 300 years ago as a look into a mathematical puzzle of whether or not a person can travel in a certain way over bridges connecting the islands of the city of Königsberg, Germany. Graph Theory has exploded in importance in the last century, both because of the growing complexity of business in a global economy and because of the computational power that computers have provided us.

Graphs

Drawing Graphs

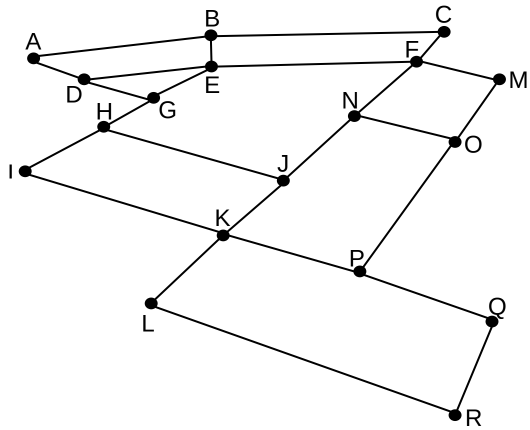
Example 1

Here is a portion of a housing development from Missoula, Montana[1]. As part of her job, the development's lawn inspector has to walk down every street in the development making sure homeowners' landscaping conforms to the community requirements.

Naturally, she wants to minimize the amount of walking she has to do. Is it possible for her to walk down every street in this development without having to do any backtracking? While you might be able to answer that question just by looking at the picture for a while, it would be ideal to be able to answer the question for any picture regardless of its complexity.

[1] Sam Beebe. http://www.flickr.com/photos/sbeebe/2850476641

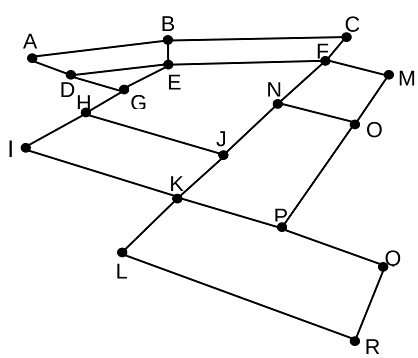
To do that, we first need to simplify the picture into a form that is easier to work with. We can do that by drawing a simple line for each street. Where streets intersect, we will place a dot.

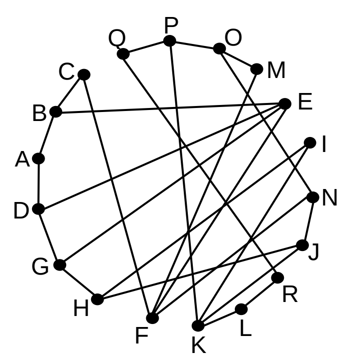


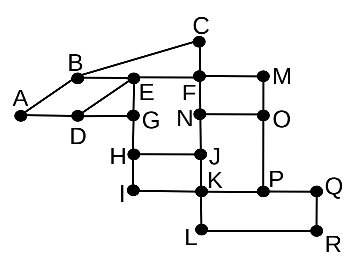
This type of simplified picture is called a **graph**. (The graph without the neighborhood is redrawn to the right and its vertices have been assigned labels for easier communication.)

> **Graphs, Vertices, and Edges**
> A **graph** consists of a set of dots, called **vertices**, and a set of **edges** connecting pairs of vertices.

While we drew our original graph to correspond with the picture we had, there is nothing particularly important about this layout when we analyze the graph. All of the graphs below are equivalent to one another since they show the same edge connections between the same pairs of vertices.



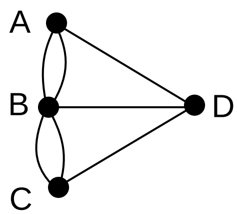




You probably already noticed that we are using the term *graph* differently than you may have used the term in the past to describe the graph of a mathematical function.

Example 2

Consider the following graph:



Notice that in the graph above there are *two* edges connecting vertices A and B as well as vertices B and C. Depending upon the interpretation of edges and vertices appropriate to a scenario, it is entirely possible and reasonable to have more than one edge connecting two vertices. (For example a mail-carrier may be asked to service the right and the left side of a street separately.)

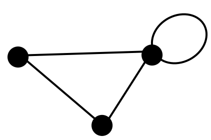
Definitions

While we loosely defined some terminology earlier, we now will try to be more specific.

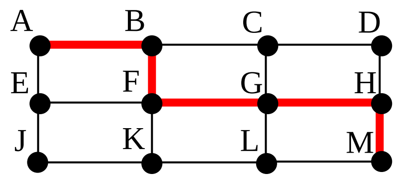
Vertex
A vertex is a dot in the graph that could represent an intersection of streets, a land mass (e.g. an island), a city, or such. Vertices are often connected by edges. Note that vertices only occur when a dot is explicitly present, not whenever two edges cross. Imagine a freeway overpass – the freeway and the street going over it cross, but it is not possible to change from one to the other at that particular point, so there is no intersection and no vertex would be placed.

Edge
An Edge connects a pair of vertices. An edge can represent a physical connection between locations, like a street, or simply that a route connecting the two locations exists, like an airline flight.

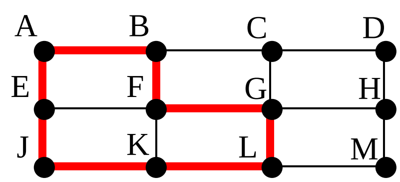
Loop
A loop is a special type of edge that connects a vertex to itself. Loops are usually not present in street network graphs, do we won't worry about them.



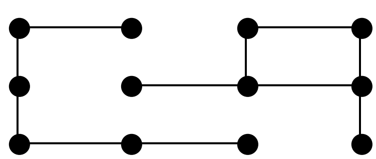
Path
A path is a sequence of vertices using existing edges. Usually we are interested in a path between two vertices. For example, a path from vertex A to vertex M is shown below. It is one of many possible paths in this graph and can be expressed as ABFGHM.



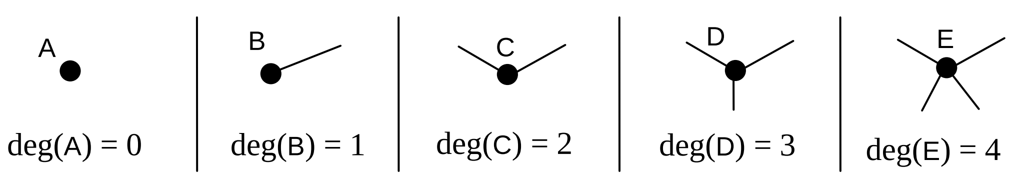
Circuit
A circuit is a path that begins and ends at the same vertex. A circuit starting and ending at vertex A is shown below. It can be expressed as ABFGLKJEA.



Connected
A graph is connected if there is a path from any vertex to any other vertex. Every graph drawn so far has been connected. The graph below is **disconnected**; there is no way to get from the vertices in the left portion to the vertices in the right portion.



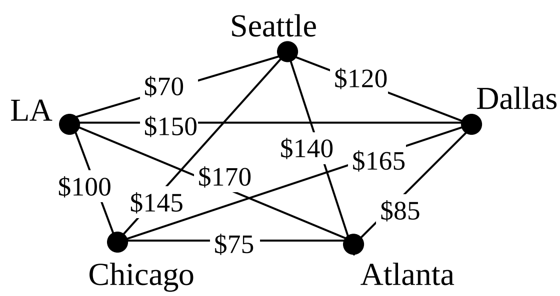
Degree of a vertex
The degree of a vertex is the number of edges meeting at that vertex.



Weight of an edge
Depending upon the problem being solved, sometimes a weight is assigned to every edge. The weights could represent the distance between two locations, the travel time, or the travel cost, etc. It is important to note that the physical distance between vertices in a graph (i.e. the length of the edge between the vertices) does not necessarily correspond to the weight of the edge.

Try it Now # 1

1. The graph below shows 5 cities. The weights on the edges represent the airfare for a one-way flight between the cities.
   a. How many vertices and edges does the graph have?
   b. Is the graph connected?
   c. What is the degree of the vertex representing LA?
   d. If you fly from Seattle to Dallas to Atlanta, is that a path or a circuit?
   e. If you fly from LA to Chicago to Dallas to LA, is that a path or a circuit?
   (*For answers, see top of next page*.)



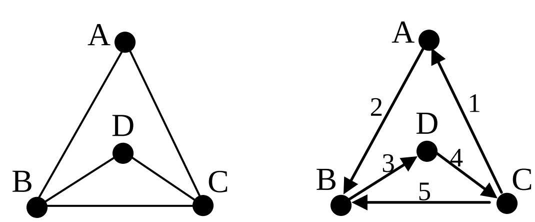
Euler Circuits

Because Euler first studied the question if a person can travel along all edges of a graph and possibly return to the starting vertex without any backtracking, paths and circuits are named after him.

Euler Path
An **Euler path** is a path that uses every edge in a graph with no repeats. Being a path, it does not have to return to the starting vertex.

Example 3

In the graph shown below, there are several Euler paths. One such path is shown to the right, starting at vertex C and ending in vertex B, with the edges numbered in the order in which they are traveled and arrows indicating the direction of each step.



ANSWERS to Try it Now # 1

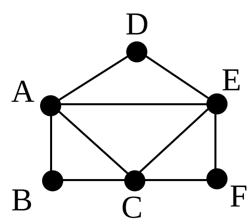
1. a. Five vertices and 10 edges
   b. Yes, the graph is connected.
   c. deg(LA) = 4
   d. Seattle to Dallas to Atlanta is a path
   e. LA to Chicago to Dallas to LA: Since you start and stop in LA, it's a circuit.

Euler Circuit

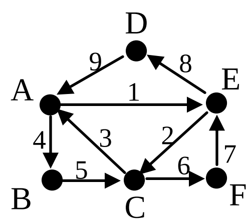
An **Euler circuit** is a circuit that uses every edge in a graph with no repeats. Being a circuit, it must start and end at the same vertex.

Example 4

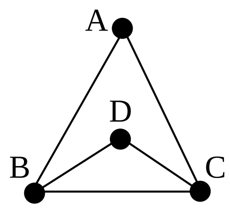
The given graph has several possible Euler circuits.



See one of them marked on the graph below.



Look at the following graph that we have considered before: Does this graph have an Euler circuit?



Any tries are failing! This graph does NOT have an Euler circuit.

Note: When weights are given, we may be interested if a shortest path exists (we will call this an optimal solution). With Euler paths and circuits, we're primarily interested in whether Euler paths or circuits ***exist***.

Why do we care if Euler circuits exist? Think back to our housing development lawn inspector from the beginning of the chapter. The lawn inspector is interested in walking as little as possible. The ideal situation would be a circuit that covers every street with no repeats (as she may want to return to a parked car – but without any backtracking). That's an Euler circuit! Luckily, Euler solved the question of whether or not Euler paths or Euler circuits will exist in a graph. His theorems are stated in the next box:

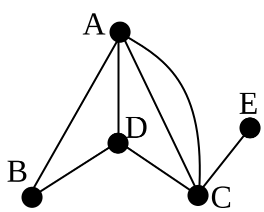
Euler's Path and Circuit Theorems

A graph will contain **Euler paths** if it contains at most two vertices of odd degree.

A graph will contain **Euler circuits** if all vertices have even degree.

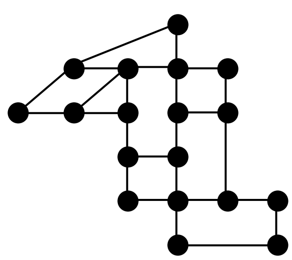
Example 5

In the graph below, vertices A and C have degree 4, since there are 4 edges leading into each vertex. Using the same reasoning, we conclude that B is degree 2, D is degree 3, and E is degree 1. This graph contains two vertices with odd degree (D and E) and three vertices with even degree (A, B, and C), so Euler's theorems tell us that this graph has Euler paths, but not Euler circuits.

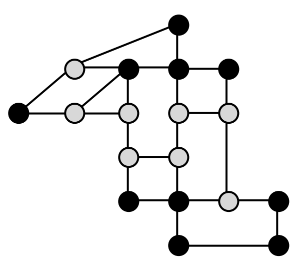


Example 6

Are there Euler circuits on the housing development lawn inspector graph we created earlier in the chapter?



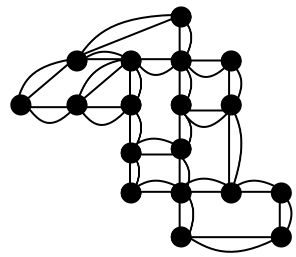
All the gray vertices marked below have odd degree.



Since there are more than two vertices with odd degree, there are no Euler paths or Euler circuits on this graph. Unfortunately our lawn inspector will need to do some backtracking.

Example 7

When it snows in the same housing development, the snowplow has to plow both sides of every street. For simplicity, we'll assume the plow is out early enough that it can ignore traffic laws and drive down either side of the street in either direction. This can be visualized in the graph by drawing two edges for each street, representing the two sides of the street.



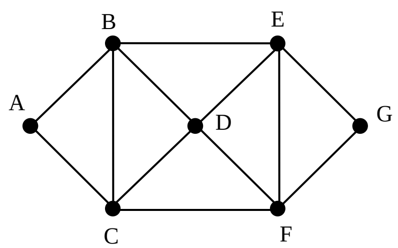
Does this graph have Euler circuits?

Notice that every vertex in this graph has even degree, so the graph above has Euler circuits.

Usually is possible to find an Euler circuit just by trying to find one and marking it up.

Try it Now # 2

Does the graph below have Euler Circuits? If so, mark one starting at A, then state the circuit by naming the vertices in the order traveled.



(*For answers, see top of next page*.)

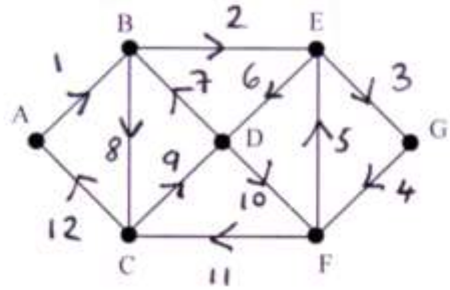
Eulerization and the Chinese Postman Problem

Not every graph has Euler paths or Euler circuits, yet our lawn inspector still needs to do her inspections. Her goal is to minimize the amount of walking she has to do. In order to do that, we will duplicate some edges in the graph until an Euler circuit exists and can be marked.

Eulerization

Eulerization is the process of duplicating existing edges on a graph to create a graph with Euler circuits. To Eulerize a graph, edges are duplicated to connect pairs of vertices with odd degree. Connecting two odd degree vertices increases the degree of each, giving them both even degree. When two odd degree vertices are not directly connected, we can duplicate all edges in a path connecting the two. A best Eulerization accomplishes this with the fewest duplicated edges.

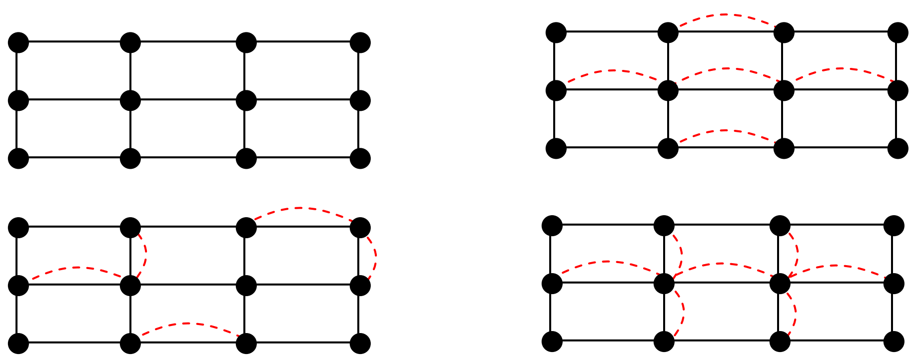
ANSWER for Try it Now # 2
Yes, the graph has Euler circuits. Euler circuits marked and stated may vary. Here is one example (starting and stopping at vertex A):



Note that we can only duplicate edges, not create an edge where there wasn't one before. Duplicating edges would mean walking or driving down a road twice, while creating an edge where there wasn't one before is like being asked to install a new road or cross private property.

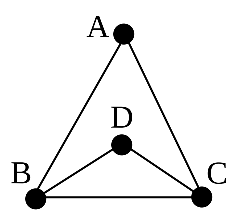
Example 8

For the rectangular graph below, three possible Eulerizations are shown. (Duplicated edges are depicted with dashed lines.) Notice in each of these cases the vertices that started with odd degrees have even degrees after Eulerization, allowing for an Euler circuit in the graph to exist. Vertices that started out with even degree are still even when the task is accomplished.



In the example above, you'll notice that the last Eulerization required duplicating seven edges, while the first two only required duplicating five edges. If we were Eulerizing the graph to find a walking path, we would want the Eulerization with minimal duplications. If the edges had weights representing distances or costs, then we would want to select the Eulerization with the minimal total added weight. However, we will not practice on graphs with weights.

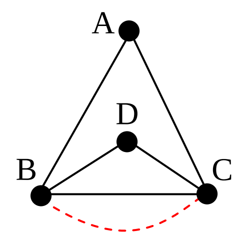
Try it Now # 3
Eulerize the graph shown.



(*For answers, see top of next page*.)

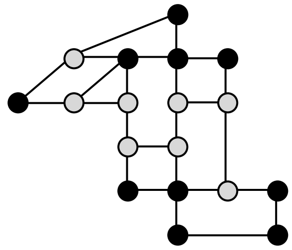
ANSWER to Try it Now # 3

The Eulerized graph includes the dashed line.

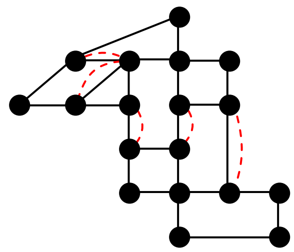


Example 9

Looking again at the graph for our lawn inspector from above, the vertices with odd degree are shown in gray.



With eight odd vertices, we will always have to duplicate at least four edges. In this case, we need to duplicate five edges (see dashed lines) since two odd degree vertices are not directly connected.



Without weights we can't be certain this is the Eulerization that minimizes walking distance, but it looks pretty good. As the fewest number of edges were duplicated, we call this a best (or optimal) Eulerization.

The problem of finding the optimal Eulerization is called the Chinese Postman Problem, a name given by an American in honor of the Chinese mathematician Mei-Ko Kwan who first studied the problem in 1962 while trying to find optimal delivery routes for postal mail-carriers. This problem is important in determining efficient routes for garbage trucks, school buses, parking meter checkers, street sweeper equipment, and more.

Unfortunately, algorithms to solve this problem are fairly complex. We will focus on simple enough graphs where it is possible to mark Euler paths and circuits and Eulerize a graph without using elaborate algorithms.