SOLUTIONS

MAT 012 Lec. Notes: ch 9.3, Absolute Value Equations and Inequalities

The **absolute value** produces **positive** outputs:

We know $|-6| = 6$

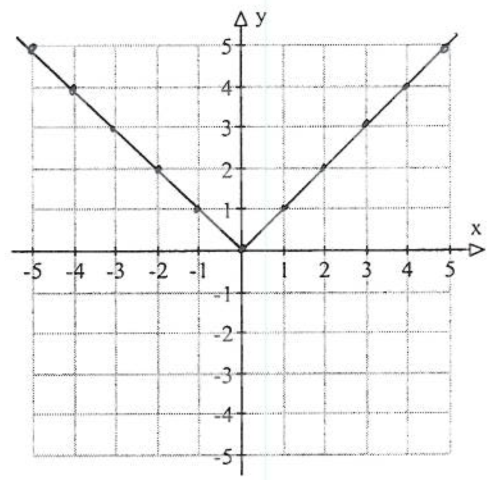
$\left|\frac{2}{3}\right| = \frac{2}{3}$

$|0| = 0$

The **definition** of the **absolute value**: $|x| = \begin{cases} x, & if\ x \geq 0 \\ -x, & if\ x < 0 \end{cases}$

The absolute value function:

x	$f(x) = \lvert x \rvert$
-5	$\lvert -5 \rvert = 5$
-4	$\lvert -4 \rvert = 4$
-3	$\lvert -3 \rvert = 3$
-2	$\lvert -2 \rvert = 2$
-1	$\lvert -1 \rvert = 1$
0	$\lvert 0 \rvert = 0$
1	$\lvert 1 \rvert = 1$
2	$\lvert 2 \rvert = 2$
3	$\lvert 3 \rvert = 3$
4	$\lvert 4 \rvert = 4$
5	$\lvert 5 \rvert = 5$



Domain:

$\mathbb{R}$

same as

$(-\infty, \infty)$

Range:

$[0, \infty)$

same as $y \geq 0$

same as $\{y \mid \geq 0\}$

Absolute Value Equations:

Give all numbers for which $|x| = 2$: Answer(s): $x = -2$ or $x = 2$

Give all numbers for which $|x| = 0$: Answer(s): $x = 0$

Give all numbers for which $|x| = \frac{1}{5}$: Answer(s): $x = -\frac{1}{5}$ or $x = \frac{1}{5}$

Give all numbers for which $|x| = -3$: Answer(s): no solutions (same as $\emptyset$)

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The **absolute value principle** for equations:

- First, isolate the absolute value expression
- Then, judge the case.

p is positive

Case 1: $\boxed{|\text{An expression with } x| = p}$

means The expression with $x = -p$ or The expression with $x = p$

Case 2: $\boxed{|\text{An expression with } x| = 0}$

means The expression with $x = 0$

Case 3: $\boxed{|\text{An expression with } x| = -p}$

Means $\varnothing$

p is positive, so $-p$ is negative

Example: Solve $|2x-3|=5$

$$2x-3=-5 \quad \text{or} \quad 2x-3=5$$

$$+3 \quad +3 \qquad\qquad +3 \quad +3$$

$$\frac{2x}{2}=\frac{-2}{2} \qquad\qquad \frac{2x}{2}=\frac{8}{2}$$

$$x=-1 \quad \text{or} \quad x=4$$

Example: Solve $|3x+10|=-2$

$$\varnothing$$

Example: Solve $\left|\frac{2x-1}{3}\right|=5$

$$\frac{2x-1}{3}=-5 \quad \text{or} \quad \frac{2x-1}{3}=5$$

$$\frac{2x-1}{3}=\frac{-5}{1} \qquad\qquad \frac{2x-1}{3}=\frac{5}{1}$$

$$2x-1=-15 \qquad\qquad 2x-1=15$$

$$+1 \quad +1 \qquad\qquad +1 \quad +1$$

$$\frac{2x}{2}=\frac{-14}{2} \qquad\qquad \frac{2x}{2}=\frac{16}{2}$$

$$x=-7 \quad \text{or} \quad x=8$$

Remember: If the absolute value expression is not isolated on one side, first isolate it and then judge the case

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Example: Solve $5-2|3x-4|=-5$ ← Note: Do not claim "no solutions" since this is only true for isolated absolute values…

Think about

$$\begin{aligned} 5-2a &= -5 \\ -5 \quad &\quad -5 \\ \hline \frac{-2a}{-2} &= \frac{-10}{-2} \\ a &= 5 \end{aligned}$$

First, isolate the absolute value.

$$\begin{aligned} 5-2|3x-4| &= -5 \\ -5 \qquad\quad &\quad -5 \\ \hline \frac{-2|3x-4|}{-2} &= \frac{-10}{-2} \\ |3x-4| &= 5 \end{aligned}$$

so

$$\begin{aligned} 3x-4 &= -5 \\ +4 \quad &\quad +4 \\ \hline \frac{3x}{3} &= \frac{-1}{3} \\ x &= -\tfrac{1}{3} \end{aligned} \quad \text{or} \quad \begin{aligned} 3x-4 &= 5 \\ +4 \quad &\quad +4 \\ \hline \frac{3x}{3} &= \frac{9}{3} \\ x &= 3 \end{aligned}$$

Example: Solve $|5x-7|-3=10$

$$\begin{aligned} |5x-7|-3 &= 10 \\ +3 \quad &\quad +3 \\ \hline |5x-7| &= 13 \end{aligned}$$

← Make sure to show this step

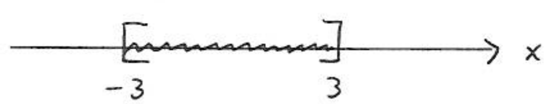
$$\begin{aligned} 5x-7 &= -13 \\ +7 \quad &\quad +7 \\ \hline \frac{5x}{5} &= \frac{-6}{5} \\ x &= -\tfrac{6}{5} \end{aligned} \quad \text{or} \quad \begin{aligned} 5x-7 &= 13 \\ +7 \quad &\quad +7 \\ \hline \frac{5x}{5} &= \frac{20}{5} \\ x &= 4 \end{aligned}$$

Note: We are skipping the problems of the type $|\text{An expression with } x| = |\text{An expression with } x|$

Absolute Value Inequalities:

Example: $|x| \le 3$, List of selected numbers that work: e.g. $-3, -2.9, -2, -\frac{3}{7}, -1, 0, 1, 2.4, 2.99,$

Now graph



so it means $-3 \le x \le 3$

Example: $|x| > 5$, List of selected numbers that work: e.g. $-2500, -7, -5.1, 5.003, 6, 7, 8, 100001$

Now graph

So it means $x < -5$ or $x > 5$ ("or" can also be displayed as $\cup$)

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In general: Again, isolate the absolute value expression, then judge the type.

"Less than" type:	$\lvert \text{An expression with } x \rvert < p$	p is positive
Solutions set-up:	$-p < \text{ The expression with } x < p$	
"Greater than" type:	$\lvert \text{An expression with } x \rvert > p$	p is positive
Solutions set-up:	The expression with $x < -p$, **or** The expression with $x > p$	

<u>Note</u>: The "less than" type is also applied for $\lvert \text{An expression with } x \rvert \le p$. Then the solution set-up uses "less than or equal to" signs.

<u>Note</u>: The "greater than" type is also applied for $\lvert \text{An expression with } x \rvert \ge p$. Then the solution set-up uses "less than or equal to" and "greater than or equal to" respectively.

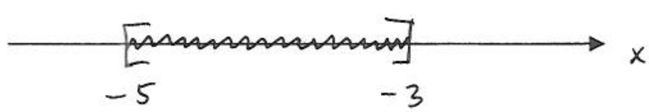
<u>Note</u>: In addition to the types mentioned above, there are some special cases, which are discussed at the end of this handout.

Example: Solve, graph on the number line and give in interval notation:

$|x+4| \le 1$ Less than type

It is correct to use $x+4 \ge -1$ and $x+4 \le 1$, but much harder to put on the number line. Most students who try it this way do <u>not</u> succeed entirely.

$$-1 \le x+4 \le 1$$
$$\underline{-4 \qquad -4 \qquad -4}$$
$$-5 \le x \le -3$$



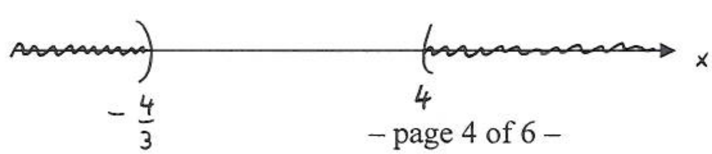
Interval notation:

$[-5, -3]$

Example: Solve, graph on the number line and give in interval notation:

$|3x-4| > 8$ greater than type

$$3x - 4 < -8 \quad \text{or} \quad 3x - 4 > 8$$
$$\underline{+4 \quad +4} \qquad\qquad \underline{+4 \quad +4}$$
$$\frac{3x}{3} < \frac{-4}{3} \qquad\qquad \frac{3x}{3} > \frac{12}{3}$$
$$x < -\frac{4}{3} \quad \text{or} \quad x > 4$$



Interval notation:

$\left(-\infty, -\frac{4}{3}\right) \cup (4, \infty)$

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Example: Solve, graph on the number line and give in interval notation:

$30-4|x+2|>12$

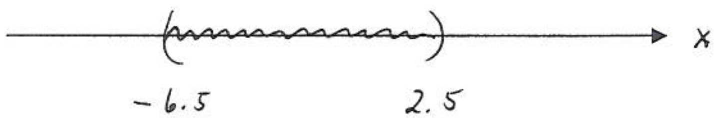
$$\begin{array}{rcl} -30 & & -30 \\ \hline \frac{-4|x+2|}{-4} & > & \frac{-18}{-4} \end{array}$$

Less than type

$|x+2| < 4.5$ same as $\frac{9}{2}$

$$\begin{array}{ccccc} -4.5 & < & x+2 & < & 4.5 \\ -2 & & -2 & & -2 \\ \hline -6.5 & < & x & < & 2.5 \end{array}$$

Same as $-\frac{13}{2} < x < \frac{5}{2}$



Interval notation:

$(-6.5, 2.5)$

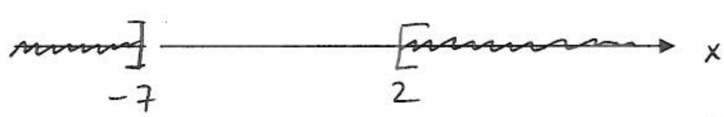
Example: Solve, graph on the number line and give in interval notation:

$|4x+10|-7 \geq 11$

$$\begin{array}{rl} & +7 \quad +7 \\ \hline & |4x+10| \geq 18 \end{array}$$

greater than type

$$\begin{array}{rcl} 4x+10 \leq -18 & \quad\text{or}\quad & 4x+10 \geq 18 \\ -10 \quad -10 & & -10 \quad -10 \\ \hline \frac{4x}{4} \leq \frac{-28}{4} & & \frac{4x}{4} \geq \frac{8}{4} \\ x \leq -7 & \quad\text{or}\quad & x \geq 2 \end{array}$$



Interval notation:

$(-\infty, -7] \cup [2, \infty)$

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Special cases of Absolute Value Inequalities

Example: $|x| > -2$ Always true (positive > negative)

$\mathbb{R}$

Example: $|x| \le -6$ Never true (positive $\le$ negative)

$\emptyset$

In general:

$\boxed{|\text{An expression with } x| > -p}$ — p is positive, so $-p$ is negative

Is always correct, since a positive number is always greater than a negative number.

Solutions: **All real numbers**.

$\boxed{|\text{An expression with } x| < -p}$ — p is positive, so $-p$ is negative

Is never correct, since a positive number can never be smaller than a negative one.

Solutions: **No solutions**.

Example: Solve $|7-2x| < -4$

$\emptyset$

Example: Solve $|x-4|+5 \ge 2$

$$\begin{array}{r} |x-4|+5 \ge 2 \\ -5 \quad -5 \\ \hline |x-4| \ge -3 \end{array}$$

$\mathbb{R}$