M 201 Lect #13 E 2-29-12

Let do some mechanics for $z = f(x,y) = xy^2 + \sin x$

① Find $\vec{\text{grad}}\, f(x,y)$

② Dir Der at (x,y)

③ Plug in $(2,1)$

④ Create a unit vector $\hat{u}$ in dir of $\langle 3,4\rangle$. $\quad \dfrac{\langle 3,4\rangle}{|\langle 3,4\rangle|}$

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① $\vec{\text{grad}}\, f = \langle f_x, f_y\rangle = \langle 1\cdot y^2 + \cos x,\ 2xy\rangle$

② $D_{\hat{u}} f = \vec{\text{grad}}\, f \cdot \hat{u} = \langle f_x, f_y\rangle \cdot \hat{u}$

$= \langle y^2 + \cos x,\ 2xy\rangle \cdot \langle u_1, u_2\rangle$

$= (y^2 + \cos x)u_1 + 2xy\, u_2$

$\hat{u} = \langle u_1, u_2\rangle = \left\langle \frac{3}{5}, \frac{4}{5}\right\rangle$

③ $D_{\hat{u}} f(2,1) = (1 + \cos 2)\, u_1 + 4u_2$

④ $D_{\hat{u}} f(2,1) \doteq (.6)\frac{3}{5} + 4\,\frac{4}{5} = \frac{1.8 + 16}{5} = \frac{17.8}{5} \doteq 3.6$

① We seek the direction in domain of steepest slope of the tan plane at a given point.

② and the slope of steepest slope. $\vec{a}\cdot\vec{b} = |\vec{a}||\vec{b}|\cos\theta$

Maximize the dir. der. Find the $\hat{u}$ that will Maximize $D_{\hat{u}} f$.

$$D_{\hat{u}} f = \overrightarrow{\text{grad}} f \cdot \hat{u} = \langle f_x, f_y \rangle \cdot \langle u_1, u_2 \rangle$$

$$= |\overrightarrow{\text{grad}} f| \cdot |\hat{u}| \cos\theta$$ ← angle between $\hat{u}$ & $\overrightarrow{\text{grad}} f$

$$= |\overrightarrow{\text{grad}} f(x_1, y_1)| \cos\theta$$

$D_{\hat{u}} f$ is max when $\cos\theta = 1$, so

" " " " $\theta = 0$

" " " " $\hat{u}$ is chosen in dir of $\overrightarrow{\text{grad}} f$

① Answer is
grad f is the dir in dom of the steepest slope of tan plane @ (x_1, y_1)

(2) Find steepest slope

$$= D_{\hat{u}} f = D_{\frac{\overline{\text{grad}}\, f}{|\overline{\text{grad}}\, f|}} f$$

$$= \overline{\text{grad}}\, f \cdot \hat{u}$$

$\vec{a} \cdot \vec{a} = |\vec{a}|^2$

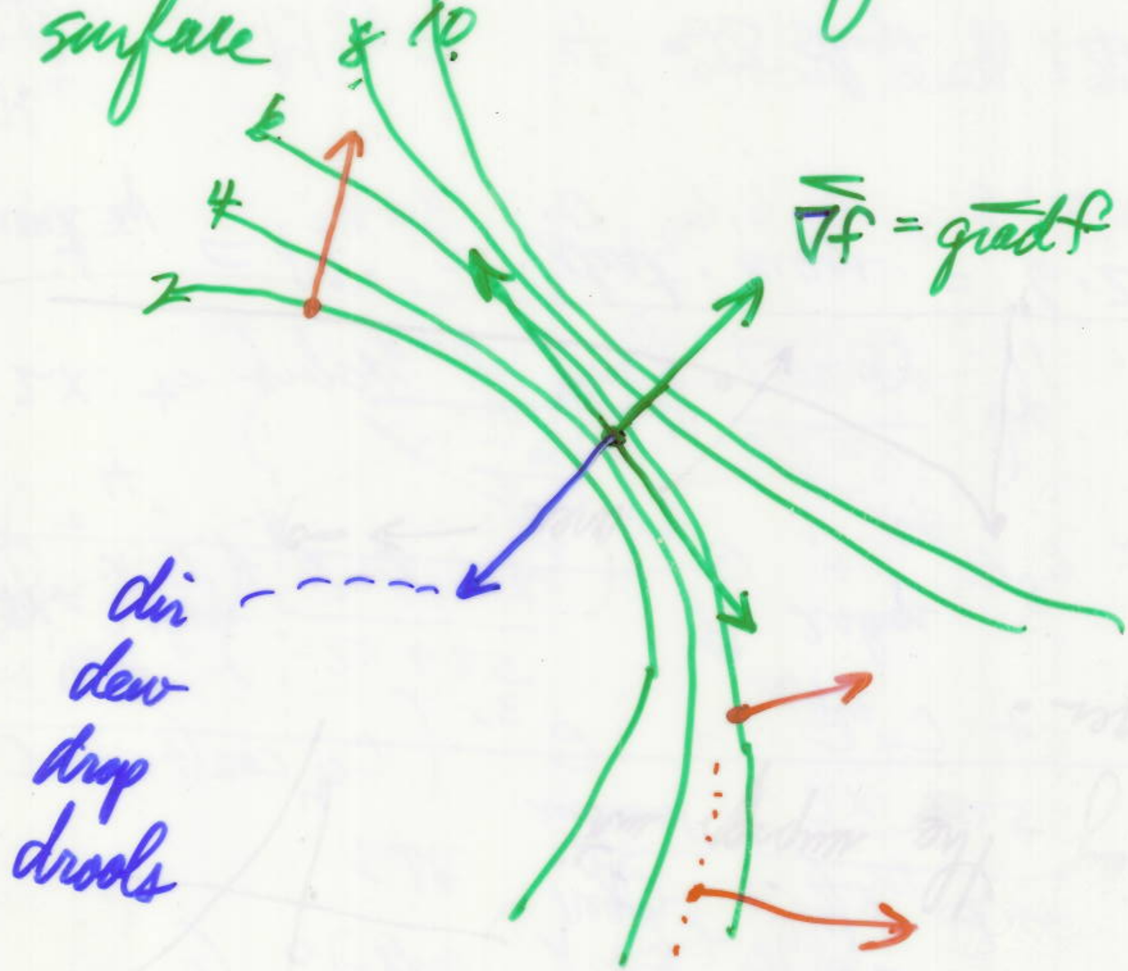
$$= \overline{\text{grad}}\, f \cdot \frac{\overline{\text{grad}}\, f}{|\overline{\text{grad}}\, f|}$$

$$= \frac{\overline{\text{grad}}\, f \cdot \overline{\text{grad}}\, f}{|\overline{\text{grad}}\, f}$$

$$= \frac{|\overline{\text{grad}}\, f|^2}{|\overline{\text{grad}}\, f|}$$

$$= |\overline{\text{grad}}\, f|$$

Suppose I draw level curves for the surface



In conclusion

The dir of steepest ascent is $\vec{\text{grad}}\, f$

The slope of steepest ascent is $|\vec{\text{grad}}\, f|$

The $\vec{\text{grad}}\, f$ on the level curves points ⊥ to level curves

Recall that we used two fcns for surfaces

$z = f(x,y)$ and $F(x,y,z) = 0$

If we have

$z = f(x,y)$, then $f(x,y) - z = 0$ so $F(x,y,z) = f(x,y) - z = 0$

For a surface like a sphere

$x^2 + y^2 + z^2 = 16$ we write $F(x,y,z) = x^2 + y^2 + z^2 - 16 = 0$

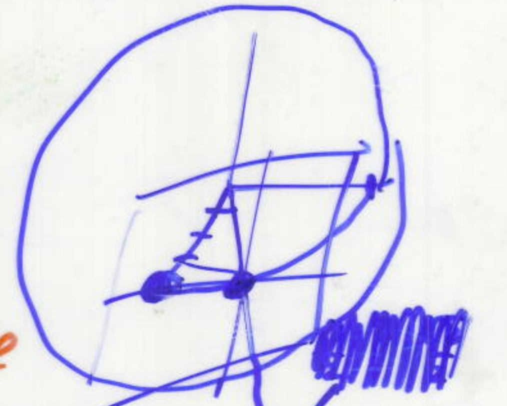
I'll show later that for a surface whose eqn is $F(x,y,z) = 0$,

$\overline{\text{grad}}\, F(x_1, y_1, z_1) \perp$ to the surface.

$\overline{\text{grad}}\, F = \langle F_x, F_y, F_z \rangle$

$\overline{\text{grad}}\, F = \langle 2x, 2y, 2z \rangle$

$\overline{\text{grad}}\, F(2, 2\sqrt{3}, 0) = \langle 4, 4\sqrt{3}, 0 \rangle \perp$ sphere



$(2, 2\sqrt{3}, 0)$ lies on the sphere.

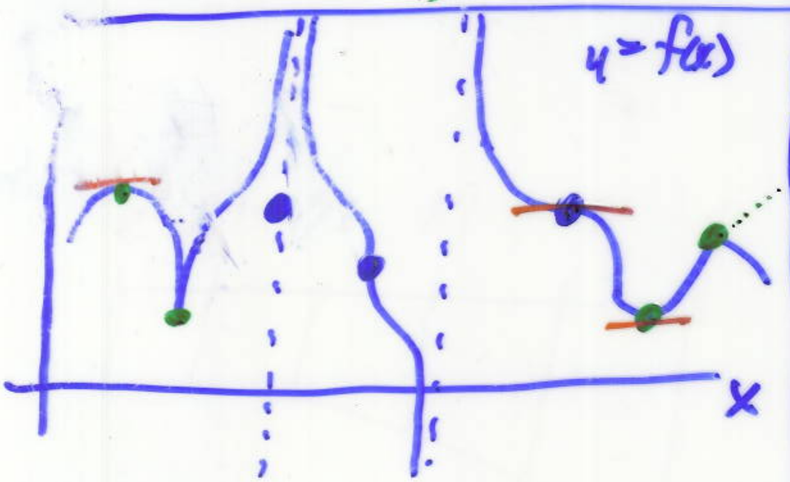
Recall eq of plane: $a(x-x_1) + b(y-y_1) + c(z-z_1) = 0$

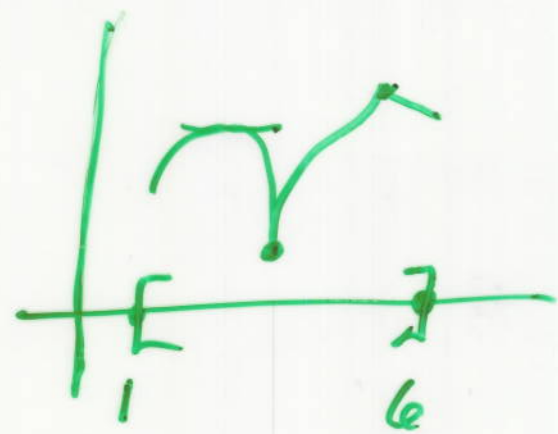
Eq of tan plane @ $\langle 4, 4\sqrt{3}, 0 \rangle$

$4(x-2) + 4\sqrt{3}(y - 2\sqrt{3}) + 0(z-0) = 0$

Recall Max/min Problems (Extrema Problems)
(Optimization Problems)
from Calc 1.

Find all relative extrema (local)	Find the absolute extrema on a close interval





I $f'(x) \overset{set}{=} 0$ x's are called critical values

II $f'(x) \overset{set}{=}$ undefined x's also called CV's

} same as here

$f''(CP)$ pos ⇒ ∪ rel. min

neg ⇒ ∩ rel max

table

	CP	y = f(x)	
EP	1	20	
CV	2	30	
CV	4	10	← abs min
CV	5	40	
EP	6	70	← abs max

Ex $z = f(x,y) = x^4 + y^4 - 4xy$

grad f $\overset{set}{=} \vec{0}$ $\langle f_x, f_y \rangle = \langle 0,0 \rangle$

$\begin{cases} f_x = 4x^3 - 4y \overset{set}{=} 0 \\ f_y = 4y^3 - 4x \overset{set}{=} 0 \end{cases}$ $\left.\begin{matrix} y = x^3 \\ x = y^3 \end{matrix}\right\}$ $x = (x^3)^3$

$x = x^9$ $(x^4)^2 - 1^2$

$x^9 - x = 0$

$x(x^8 - 1) = 0$

$x(x^4+1)(x^4-1) = 0$

$x(x^4+1)(x^2+1)(x^2-1) = 0$

$x(x^4+1)(x^2+1)(x+1)(x-1) = 0$

$x = 0, -1, 1$ $y = 0, -1, 1$ resp.

So CP's are $(0,0)$, $(-1,-1)$ $(1,1)$

No Type II CP's

We create the Discriminant

$f_x = 4x^3 - 4y$ $f_y = 4y^3 - 4x$

$f_{xx} = 12x^2$ $f_{xy} = -4$ $f_{yx} = -4$ $f_{yy} = 12y^2$

$$Disc(x,y) = \begin{vmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{vmatrix} = f_{xx}f_{yy} - f_{xy}^2$$

$$= \begin{vmatrix} 12x^2 & -4 \\ -4 & 12y^2 \end{vmatrix} = 144x^2y^2 - 16$$