<u>Bell's Theorem</u>: For any <u>lattice polygon</u> in the plane (each vertex is a <u>lattice point</u>, i.e. has integer coordinates), let:

> **I** = # of lattice points inside the polygon = #(Int)
> **B** = # of lattice points on the boundary of the polygon = #(Bndy)

then **Area = I + B/2 – 1** .

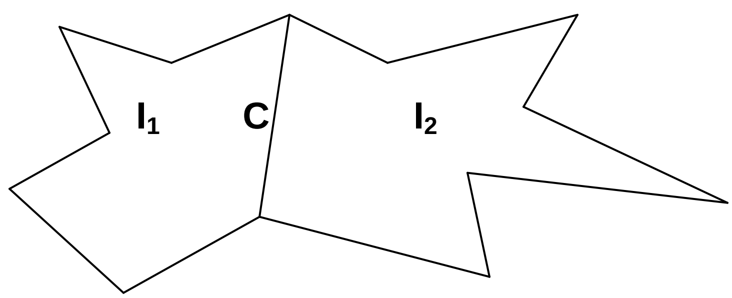
Proof will be via strong induction on number of sides, **N ≥ 3**. Do **N=3**, a lattice triangle, first … and …we'll need a Lemma …..

Lemma: Let a lattice polygon be divided into two other lattice polygons with a common line having **C ≥ 0** lattice points (not counting vertices). Let:

A = Area of whole,	A_1 = Area of piece 1,	A_2 = Area of piece 2,
I = #(Int in whole),	I_1 = #(Int piece 1),	I_2 = #(Int piece 2),
B = #(Bndy on whole),	B_1 = #(Bndy piece 1),	B_2 = #(Bndy piece 2)

Then (i) $A = A_1 + A_2$ (obviously, since areas are additive)

(ii) $(I + B/2 - 1) = (I_1 + B_1/2 - 1) + (I_2 + B_2/2 - 1)$



Proof: $I - C = I_1 + I_2$, since removing the lattice points on the common line leaves all the rest of the interior lattice points of the whole, which is the sum of the interior points of the two pieces.

$B = B_1 + B_2 - 2C - 2$, since the common line is not on the boundary of whole and the two end points of the common line are counted twice, **C** must be subtracted twice and **2** must be subtracted once.

Compute: $I_1 + B_1/2 - 1 + I_2 + B_2/2 - 1 = (I_1 + I_2) + (B_1 + B_2)/2 - 2 =$
$(I - C) + (B + 2C + 2)/2 - 2 = I - C + B/2 + 2C/2 + 2/2 - 2 =$
$I + B/2 - 1.$ (qed)

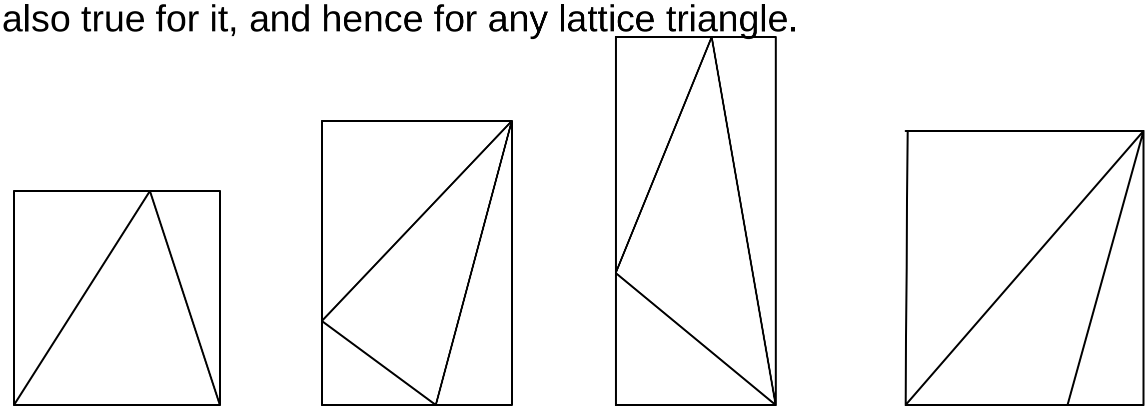
Corollary: If Bell is true for two of the three pieces in the Lemma, it's true for the third.

Proof of Theorem: First consider a right triangle with two sides parallel to the axes, which is half of a rectangle with sides parallel to the axes.

Let a lattice rectangle have sides parallel to the axes. We may assume the corners are: (0,0), (n,0), (0,m), (n,m). So **A** = nm; **I** = (n-1)(m-1); **B** = 2(n+1) + 2(m-1) = 2(n+m); thus **I + B/2 – 1** = (n-1)(m-1) + 2(n+m)/2 – 1 = nm – n – m + 1 + n + m – 1 = nm = **A** . So, Bell is true.

Such a parallel lattice right triangle is half of such a lattice rectangle. Using the notation of the Lemma: $\mathbf{A = 2A_1}$ and $\mathbf{I + B/2 - 1 = 2(I_1 + B_1/2 - 1)}$; and, since Bell is true for **A**, it's true for $\mathbf{A_1}$.

Any lattice triangle may be surrounded by lattice right triangles inside a lattice rectangle, as shown. A piece by piece analysis shows that since Bell is true for every piece except the central, original lattice triangle, it is also true for it, and hence for any lattice triangle.



So, Bell is true for **N=3**.

Assume Bell is true for all lattice polygons with number of sides **≤ N** . Consider a lattice polygon with **N+1** sides. It may be divided into two lattice polygons with **≤ N** sides using a common line as in the Lemma. By the strong induction hypothesis, Bell is true for each piece and hence, by the lemma, Bell is true of the whole. (qed)

(Are there any problems with this proof? Can we always divide one lattice polygon into two by adding an interior line?)