Probable Probability

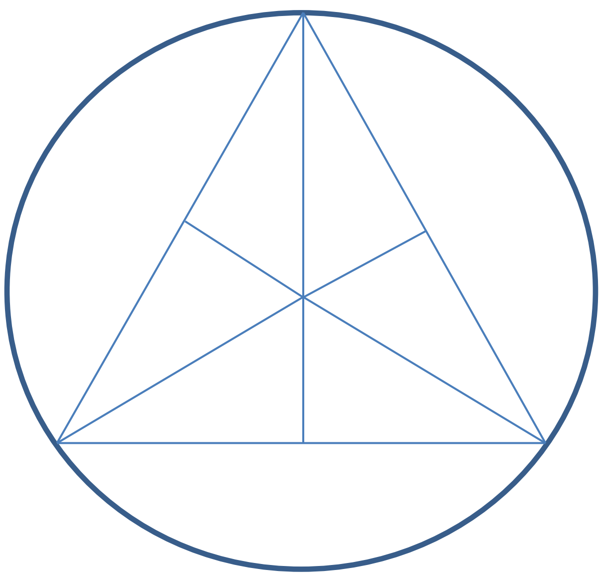
Abstract: This dramatization is based upon an actual Calculus II class. As is not uncommon, the students took unplanned control of a class and I merely moderated. The names of both the innocent and the guilty have been disguised in homage to Imra Lakatos's classic "Proofs and Refutations".

Professor Pi (Entering the classroom to find geometric diagrams on the blackboard and the students engaged in a vehement discussion.) What's going on here?

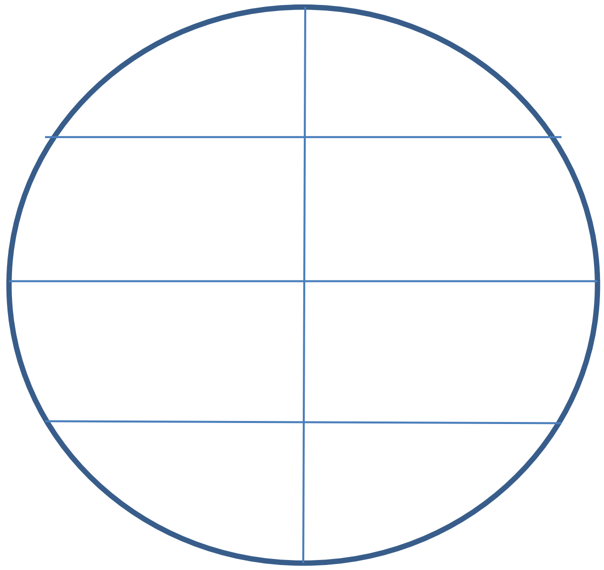
Beta: It's that probability question you assigned on Friday. In a circle of radius **R**, a chord is chosen at random. What is the probability that its length is greater than the side of the inscribed equilateral triangle? Alpha, Gamma and Delta have three different answers. They all seem correct, but that can't be right.

Professor Pi: Interesting. What is the length of the side?

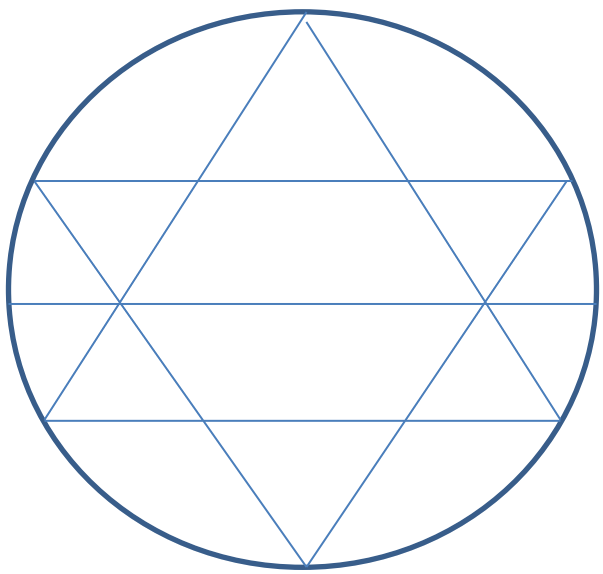
Beta: That's easy. Just divide the equilateral triangle into six 30-60-90 triangles, each with hypotenuse **R** and short-side $^{R}/_{2}$. The altitude of the equilateral triangle must be $^{3R}/_{2}$ and the side is then $\mathbf{R\sqrt{3}}$. That's not what the dispute is. Explain your answer, Alpha.



Alpha: A chord is determined by its endpoints, so pick two points on the circle at random. By symmetry, we can rotate the circle and assume the points are on the same horizontal level. So choosing a chord is the same as choosing a point along the bisecting diameter, which we can assume is vertical. Using a uniform distribution on the diameter $\mathbf{[-R,R]}$, the chord has length longer than $\mathbf{R\sqrt{3}}$ if the point is in $\mathbf{[^{-R}/_{2},{}^{R}/_{2}]}$, which gives a probability of $^{R}/_{2R} = {}^{1}/_{2}$.

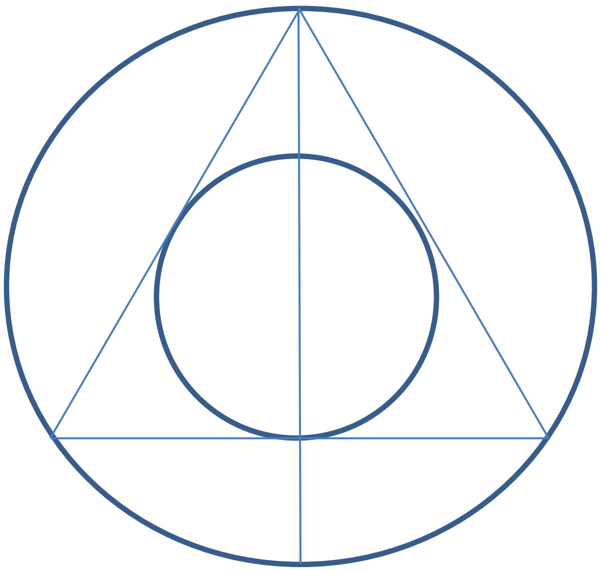


Gamma: I picked the endpoints, too, but I did it differently. The first one doesn't matter, so we can rotate the circle to put it at the top. The second point can be picked anywhere along the circle, so use a uniform distribution on **$[0,2\pi R]$** . To be longer than **$R\sqrt{3}$** , the point must be on one of the two arcs on either side of the middle horizontal diameter**.** The two excluded arcs are both $^1/_3$ of the circle, so the sum of the two middle arcs is also – you can see it from Beta's diagram of the equilateral triangle – which gives a probability of $^1/_3$.



Professor Pi: That is interesting. Both your methods make perfectly good sense, too. Beta said there's a third answer. What was yours, Delta?

Delta: Both Alpha and Gamma had to rotate the circle and I think that might distort the probability. I wanted to pick a chord without having to rotate it. With the exception of the center, every point inside a circle determines a unique chord for which it is the midpoint. So to pick a chord, we need to just pick a point inside the circle. The total area is πR^2. The chord so determined has length greater than $R\sqrt{3}$ if the point is in the central circle of radius $R/2$, with area $\pi(R/2)^2$, which gives a probability of $1/4$.



Professor Pi: Very clever.

Delta: I think mine is the right answer. The picture clearly shows the set of points corresponding to the longer chords is much smaller than half the circle.

Professor Pi: Your answer is the correct answer, Delta. But so are Alpha's and Gamma's.

Beta: That's not possible.

Professor Pi: But it is. This problem is actually quite famous. It is called Bertrand's Paradox.[1] I am quite impressed that you found the same three different approaches to the problem that Bertrand found. It is not actually a paradox. What was our definition of a probability space? Beta? You take excellent notes, why don't you write it on the blackboard?

Beta: A probability space is a set $\mathbf{S} = \{$outcomes of some random process$\}$, along with a function, $\mathbf{P} : \mathbf{S} \rightarrow \mathbf{R}^+$, the positive real numbers.

Professor Pi: Very good. And to calculate probabilities?

Beta: **P** gives the probability of an outcome. An event is a subset of **S**. The probability of an event is the sum of finite probabilities if S is finite, or, the integral of the infinitesimal probabilities if S is continuous. For a continuous uniform **P** , just compute length or area or volume.

Professor Pi: Excellent, Beta. Notice a random process is part of the definition. Each of you has calculated the correct probability, but for a different random process. Alpha, you chose the two endpoints of the chord simultaneously. Gamma, you chose the two endpoints sequentially. Delta, you chose the midpoint of the chord. Different random processes yield different probabilities.

Delta: But isn't there a right way to choose the chord completely randomly?

Professor Pi: Some philosophers have argued that there is. I really do not think so. What is the correct way to randomly choose three cards from a standard 52-card deck? With or without replacement?

Beta: Well, you can't deal cards with replacement.

Professor Pi: Very true, Beta. But you do not have to deal three cards to choose three. Different processes yield different random choices. Delta, your method might be to choose a chord by throwing a dart at the circle while blind-folded. Gamma, your method might be to put a spinner at the center of the circle and spin it twice to choose the endpoints. Alpha, your method might be to drop along stick onto the circle from a great height without putting a spin on the stick. They each yield different probabilities and this is not really surprising.

Beta: I like it better when there is one right answer.

Professor Pi: If life was really that simple, Beta, it would be a whole lot less interesting.

References:

[1] Joseph L. F. Bertrand, Calcul des Probabilites (Paris: Gauther-Villars, 1889)