Algebra Basics

Laws of Algebra
FOIL, Factoring and Expanding
Standard Factor Rules

$$A^2 - B^2 = (A + B)(A - B)$$
$$(A \pm B)^2 = A^2 \pm 2AB + B^2$$
$$A^3 \pm B^3 = (A \pm B)(A^2 -\pm AB + B^2)$$
$$(A \pm B)^3 = A^3 \pm 3A^2B + 3AB^2 \pm B^3$$

Factorial: n! is number of ways to arrange n "guys" in an order.

$$0! = 1 \; ; \; 1! = 1 \; ; \; n! = n\cdot((n-1)!) = n\cdot(n-1)\cdot(n-2)\cdot\ldots\cdot 2\cdot 1$$

Binomial Theorem:

$$(A+B)^n = \sum_{k=0}^{n} {}_nC_k \, A^k B^{n-k} \quad \text{, where}$$

$${}_nC_k = \frac{n!}{k!(n-k)!}$$ **, number of ways to choose k guys from n guys.**

Polynomials and Rational Expressions: Operations

Arithmetic similar to "long-hand" methods for decimal numbers
Add/Subtract Fractions (find common denominator)
Clear Compound Fractions (multiply by reciprocals)

Negative Exponents are Reciprocals
Fractional Exponents are Radicals

Clear radical sums using conjugates: $(x + y)(x - y) = x^2 - y^2$

Solving Equations and Inequalities (Always check − plug back in)

"Both Sides" Rules
- **(B.S.) + C**
- **(B.S.) · C**
- **(B.S.)^C = $(B.S.)^C$**
- $\pm\sqrt{(B.S.)}$
- **Inverse function(B.S.)**

One Variable Strategies:

All variables on one side, numbers on other side
Clear fractions (reciprocal) and Clear radicals (squaring)
Set = 0, Factor and Split
Inequalities - solve equation and test regions

Quadratic Formula: (Completing the Square)

If $ax^2 + bx + c = 0$ then $x = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$

Number of real solutions and **Discriminant ($b^2 - 4ac$):**
> 0, two solutions; **= 0**, one solution, **< 0**, no solution

Word Problems

Read, **ReRead**, and find applicable model
Unknowns are variables; Knowns yield equations
Solve the equations
Check for accuracy and reasonableness!

Equations, Functions and Graphs

Cartesian Coordinates, (x, y), for the plane
Distance between points (x_1, y_1) and (x_2, y_2) : $\sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$
Midpoint between points: $\left(\frac{x_2 + x_1}{2}, \frac{y_2 + y_1}{2} \right)$

Functions - a well-defined rule, **y = f(x),** relating elements of two sets.
Domain = { all x where f(x) is defined} **Range** = { all y = f(x) for some x }
Graphing Functions - satisfies **Vertical Line Rule -** Plot sample values
X-intercepts (solve **f(x)=0**) and **Y-intercepts (** evaluate **f(0))**

Lines and Linear Functions
General linear equation **Ax + By = C**
Slope-Intercept formula: **y = f(x) = mx + b** for non-vertical lines
Slope $m = \Delta y / \Delta x = (y_2 - y_1) / (x_2 - x_1)$
Y-intercept $y = f(0) = b = y_1 - mx_1$
X-intercept **x = -b/m.**
To find the equation of line through two points, calculate **m** and **b**

Transformations and Combinations of Functions
Given two functions, **f(x) and g(x)**, may define on appropriate domains:
$(f+g)(x) = f(x) + g(x)$ $(f-g)(x) = f(x) - g(x)$
$(f\times g)(x) = f(x) \times g(x)$ $(f/g)(x) = f(x) / g(x)$
$(g\circ f)(x) = g(f(x))$ – **composition** of **f(x)** and **g(x)**.
Given a function, **y = f(x)**, may define transformations: **y = Af(ax + b) + B**:
A stretches graph vertically, with Y-int fixed
a reverse stretches graph horizontally, with X-int fixed
B shifts graph vertically **b** reverse shifts graph horizontally
A **1-1** function, **f(x)**, (Horizontal Line Rule) has an inverse function, $f^{-1}(x)$
$(f^{-1}\circ f)(x) = x = (f\circ f^{-1})(x)$
To find f^{-1}, solve **y = f(x)** for **x (and swap x&y)**

Parabolas and Quadratic Functions
Equation $y = f(x) = ax^2 + bx + c = a(x - h)^2 + k$, for $a \neq 0$
Shape determined by **|a|: steep** if |a| big, **flat** if |a| small
concave up (minimum) if **a>0** **concave down (maximum)** if **a<0**.
Vertex of parabola is **(x,y) = (-b/2a, f(-b/2a)) = (h,k)**
Y-intercept **= c.** **X**-intercepts = (**Quadratic Formula**) = $\frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$

Conic Sections centered at origin (0,0):
Circle: $x^2 + y^2 = r^2$ (radius=**r**)

Ellipse: $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$
X-int = ±**a**, Y-int = ±**b**

Hyperbola: $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$
Asymptotes: $y = (\pm {}^{b}/_{a})x$

Polynomial and Rational Functions

Polynomial functions: $y = P(x) = a_n x^n + a_{n-1}x^{n-1} + ... + a_1 x + a_0$

Behavior mostly determined by **degree** (**n**) and **leading term** ($a_n x^n$).
Roots or **x-intercepts, x = r,** correspond to **factors, (x – r),** which divide **P(x)**.
Division of polynomials: $P(x) = D(x)*Q(x) + R(x)$; or $\frac{P(x)}{D(x)} = Q(x) + \frac{R(x)}{D(x)}$.

	Odd Degree, $a_n>0/a_n<0$	Even Degree, $a_n>0/a_n<0$
Domain:	{ All x }	{ All x }
Range:	{ All y }	$\{ y \geq MIN \} / \{ y \leq MAX \}$
y-int:	$P(0) = a_0$	$P(0) = a_0$
x-int:	At least 1, at most n	Maybe 0, at most n
Bumps:	Maybe 0, at most n-1	At least 1, at most n-1
$x \to +\infty$:	$y \to +\infty / -\infty$	$y \to +\infty / -\infty$
$x \to -\infty$:	$y \to -\infty / +\infty$	$y \to +\infty / -\infty$

Complex Numbers: **a + bi,** where **a,b** are real numbers and $i^2 = -1$.

The usual operations, { +,-,×,/ } follow all the usual **Laws of Algebra.**

Remember that division uses the conjugate (**a + bi** and **a - bi** are conjugates).

Fundamental Theorem of Algebra (Gauss): A polynomial of degree **n** has **n** roots.

(Assuming complex roots are allowed and roots are counted with multiplicity.)

Rational functions: $y = f(x) = \frac{P(x)}{Q(x)} = \frac{a_n x^n + a_{n-1}x^{n-1} + ... + a_1 x + a_0}{b_m x^m + b_{m-1}x^{m-1} + ... + b_1 x + b_0}$

Assume any common factors in **P(x)** and **Q(x)** are cancelled.
Behavior mostly determined by **degrees** (**n** & **m**) and **leading terms** ($a_n x^n$ & $b_m x^m$).

Domain:	$\{ x \mid Q(x) \neq 0 \}$, exclude roots of Q(x)
Range:	Complicated by Asymptotes and End Behavior
y-int:	$R(0) = a_0 / b_0$
x-int:	Roots of P(x) - at most n
Local Max/Min:	"Bumps" - Maybe 0, at most n+m-1
Global Max/Min:	May not be one: see asymptotes

Vertical Asymptote(VA): **x = r at each root, r, of Q(r) (** factors are **Q(x))**

Asymptotes as as $x \to \pm\infty$:

Horizontal(HA):	$y = 0$ if $n < m$
	$y = a_n/b_n$ if $n = m$
Slant (SA):	y = quotient after divide Q(x) into P(x)
None:	$y \to \pm\infty$ if $n > m+1$

Exponential and Logarithmic Functions

Exponential Functions: $y = a^x$, where the base $a > 0$ and $a \neq 1$

$y = a^{-x} = (1/a)^x = 1/a^x$

	a > 1	**1 > a > 0**
Domain:	{ all x }	{ all x }
Range:	{ y > 0 }	{ y > 0 }
y-int:	(0,1)	(0,1)
x-int:	none	none
1-1 monotone	increasing	decreasing
$x \to +\infty$	$y \to +\infty$ fast	$y \to 0$ (HA)
$x \to -\infty$	$y \to 0$ (HA)	$y \to +\infty$ fast

To graph transformations: plot >3 points; determine HA-asymptotoes as $x \to \pm\infty$

Algebraic Rules: $a^x a^y = a^{x+y}$; $(a^x)^y = a^{xy}$; $a^0 = 1$; $a^{-x} = 1/a^x$

Applications: Annual Growth at **r%:** $A(t) = A(0)(1 + r)^t$

Continuous Growth at **r%:** $A(t) = A(0)e^{rt}$

where $e = \lim (1 + {}^1/_n)^n \approx 2.71828....$

Population Growth, Doubling Time, **D**: $A(t) = A(0)2^{t/D}$

Radioactive Decay, Half-Life, **H**: $A(t) = A(0)2^{-t/H}$

Compound interest at **r%** :

compounded m times per year: $B(t) = B(0)(1 + r/m)^{mt}$

compounded continuously: $B(t) = B(0)e^{rt}$

Logarithmic Functions: $y = \log_a(x)$, where base $a > 0$ and $a \neq 1$

(the log is the exponent: $\log_a(a^x) = x$ or $y = \log_a(x)$ means $a^y = x$)

	a > 1	**1 > a > 0**
Domain:	{ x > 0 }	{ x > 0 }
Range:	{ all y }	{ all y }
y-int:	none	none
x-int:	(1,0)	(1,0)
1-1 monotone	increasing	decreasing
$x \to +\infty$	$y \to +\infty$ slow	$y \to 0$ (VA)
$x \to 0$	$y \to 0$ (VA)	$y \to +\infty$ slow

To graph transformations: plot >3 points; determine VA-asymptotes at “edge” of domain.

Algebraic Rules: $\log_a(xy) = \log_a(x) + \log_a(y)$ $\log_a(x^y) = y\log_a(x)$

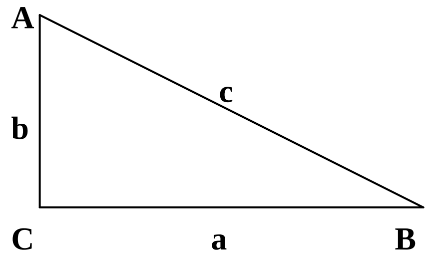
Change of base: $\log_a(x) = \dfrac{\log(x)}{\log(a)}$ $\log_a(1) = 0$; $\log_a(a) = 1$

Common Log: $\log_{10}(x) = \log(x)$ **Natural Log:** $\log_e(x) = \ln(x)$

Applications: pH = -log([H+]); Richter Scale; Decibels; Orders of Magnitude

Trigonometry and Triangles

Right Triangles:



Angle C = 90°
Angles A + B = 90°
Sides: $c^2 = a^2 + b^2$
(Pythagorus)

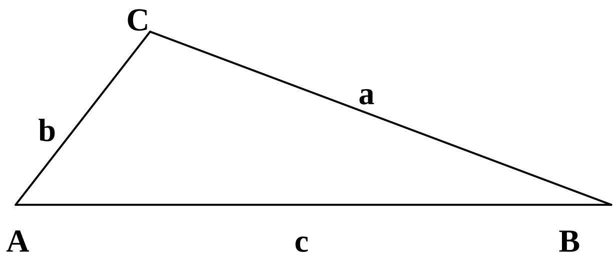
Sine:	**sin(A) = a/c = cos(B)**	**sin(A) = sin(180-A)**
Cosine:	**cos(A) = b/c = sin(B)**	**cos(A) = -cos(180-A)**
	sin(0°) = 0 = cos(90)	**sin(90°) = 1 = cos(0)**

Tangent:	**tan(A) = a/b = cot(B) = sin(A)/cos(A) = cos(B)/tan(B)**
Cotangent:	**cot(A) = b/a = tan(B) = cos(A)/sin(A) = sin(B)/cos(B)**
Secant:	**sec(A) = c/b = csc(B) = 1/cos(A) = 1/sin(B)**
Cosecant:	**csc(A) = c/a = sec(B) = 1/sin(A) = 1/cos(B)**

Area: $ab/2 = ac\times\sin(B)/2 = bc\times\sin(A)/2$
Pythagorean: $1 = \sin^2(A) + \cos^2(A) = \sec^2(A) - \tan^2(A) = \csc^2(A) - \cot^2(A)$

Right Isosceles Triangle: $\sin(45°) = \cos(45°) = \sqrt{2}/2 \approx 0.707...$
Half Equilateral Triangle: $\sin(30°) = \cos(60°) = 1/2 = 0.5$
$\sin(60°) = \cos(30°) = \sqrt{3}/2 \approx 0.866...$

All Triangles:
(Right, Acute or Obtuse)



Angles: A + B + C = 180°

Area: $A = ab\times\sin(C)/2 = ac\times\sin(B)/2 = bc\times\sin(A)/2$
(Heron) $A = (s(s-a)(s-b)(s-c))^{1/2}$, **where** $s = (a+b+c)/2$

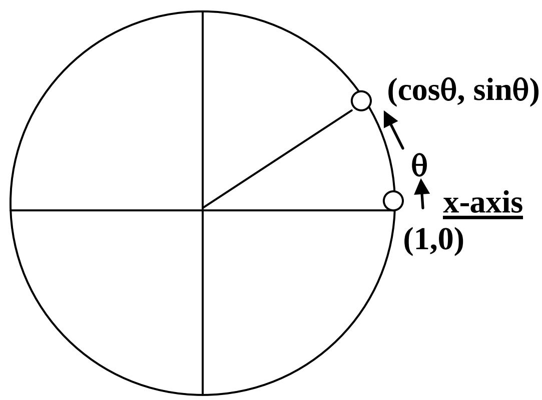
Law of Sines: $\sin(A)/a = \sin(B)/b = \sin(C)/c$
Law of Cosines: $a^2 = b^2 + c^2 - 2bc\times\cos(A)$
$b^2 = a^2 + c^2 - 2ac\times\cos(B)$
$c^2 = a^2 + b^2 - 2ab\times\cos(C)$

Applications: **"Solving" Triangles – Surveying, Navigation, Vectors, etc**
Similar Δ's: **Angles equal (sides proportional): AAA**
Congruent Δ's: **Sides & angles equal: SSS, SAS, ASA, AAS (not ASS)**

Trigonometry, Circles and Functions

Unit Circle: $x^2 + y^2 = 1$

Angle, θ, in radians = length along arc from (1,0) to ($\cos\theta$, $\sin\theta$)



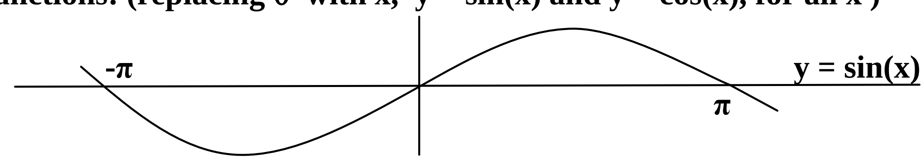
$\theta > 0$ counter-clockwise, $\theta < 0$ clockwise

(angle in rad) = $(\pi/180)\times$(angle in degrees)

$90° = \pi/2$ rad (from (1,0) to (0,1))

Co-terminal if = $\pm 2\pi$ or $\pm 360°$

Trig functions: (replacing θ with x, y = sin(x) and y = cos(x), for all x)



Transformations: **$y = A\sin(k(x-p)) + B$** **Amplitude = $|A|$**

period = $2\pi/k$ = 1/Frequency **phase-shift = p**

$y = \cos(x) = \sin(\pi/2 - x)$

$\tan(x) = \sin(x)/\cos(x)$, $\sec(x) = 1/\cos(x)$, for $x \neq \pm k\pi + \pi/2$, k=0,1,2,....

$\cot(x) = \cos(x)/\sin(x)$, $\csc(x) = 1/\sin(x)$, for $x \neq \pm k\pi$, k=0,1,2,....

Periodic:	$\sin(x) = \sin(x+2\pi)$	$\cos(x) = \cos(x+2\pi)$	$\tan(x) = \tan(x+\pi)$
Cofunctions:	$\sin(\pi/2 - x) = \cos(x)$	$\cos(\pi/2 - x) = \sin(x)$	$\tan(\pi/2 - x) = \cot(x)$
Odd/Even:	$\sin(-x) = -\sin(x)$	$\tan(-x) = -\tan(x)$	$\cos(-x) = \cos(x)$

Pythagorus: $1 = \sin^2(x) + \cos^2(x) = \sec^2(x) - \tan^2(x) = \csc^2(x) - \cot^2(x)$

Sum angles:

$\sin(u + v) = \sin(u)\cos(v) + \sin(v)\cos(u)$

$\cos(u + v) = \cos(u)\cos(v) - \sin(u)\sin(v)$

$\tan(u+v) = (\tan(u) + \tan(v)) / (1 - \tan(u)\tan(v))$

Inverse functions:

$y = \sin^{-1}(x) = \arcsin(x)$, for $x \in [-1,1]$, $y \in [-\pi/2, \pi/2]$, $\sin(y) = x$

$y = \cos^{-1}(x) = \arccos(x)$, for $x \in [-1,1]$, $y \in [0, \pi]$, $\cos(y) = x$

$y = \tan^{-1}(x) = \arctan(x)$, for $x \in (-\infty,\infty)$, $y \in (-\pi/2, \pi/2)$, $\tan(y) = x$

Applications: **Circular motion, Harmonic motion, Wave motion**