Limit – "Where IT Wants to Go"

Finite limits are one of: $x \to a$; $x \to a^+$; $x \to a^-$

To find: lim f(x) plug in "a" and compute f(a):

1. **If f(a) = 0 , # , 0/# = 0, for # ≠ 0 : Done (continuous)**

2. **If f(a) = #/∞ = 0/∞ = 0 : Done**

   If f(a) = ∞ = ∞/# = #/0 = ∞/0 : Done (Arithmetic of ∞ = ±∞)

3. **If f(a) = 0/0 , ∞/∞ , ∞ − ∞ , 0·∞ , 0^0 , 1^∞ or ∞^0 : Not done**
   (These are the **7** Indeterminate Forms. Must do something clever.)

3.a. **Fiddle** **(Factor, common denominator, radical conjugates, etc.)**

3.b. **Squeeze Rule: If $f(x) \le g(x) \le h(x)$**
then $\lim f(x) \le \lim g(x) \le \lim h(x)$

3.c. **L'Hospital's Rule:**

If $\lim \dfrac{f(x)}{g(x)} = \dfrac{0}{0}$ or $\dfrac{\infty}{\infty}$ & $\lim \dfrac{f'(x)}{g'(x)}$ **exists**

then $\lim \dfrac{f(x)}{g(x)} = \lim \dfrac{f'(x)}{g'(x)}$.

Infinite limits are one of: $x \to \infty$; $x \to +\infty$; $x \to -\infty$

To find: May let x = 1/t with t → 0 and fiddle or use L'Hospital.

Important limits

$$\lim_{x\to 0} \frac{\sin(x)}{x} = 1 \; ; \qquad \lim_{x\to 0} \frac{e^x - 1}{x} = 1 \; ; \qquad \lim_{x\to\infty} x^n e^{-n} = 0$$

Derivative – Rate of Change – Slope of Tangent

$$f'(x) = y' = y'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = \frac{dy}{dx}$$

Elementary Functions:

$$\frac{d\,x^n}{dx} = nx^{n-1} \qquad \frac{d\,e^{ax}}{dx} = ae^{ax} \qquad \frac{d\ln(x)}{dx} = \frac{1}{x}$$

$$\frac{d\sin(ax)}{dx} = a\cos(ax) \qquad \frac{d\tan(ax)}{dx} = a\sec^2(ax) \qquad \frac{d\sec(ax)}{dx} = a\sec(ax)\tan(ax)$$

$$\frac{d\cos(ax)}{dx} = -a\sin(ax) \qquad \frac{d\cot(ax)}{dx} = -\,a\csc^2(x) \qquad \frac{d\csc(ax)}{dx} = -a\csc(ax)\cot(ax)$$

$$\frac{d\sin^{-1}(x)}{dx} = \frac{1}{\sqrt{1-x^2}} \qquad \frac{d\tan^{-1}(x)}{dx} = \frac{1}{1+x^2} \qquad \frac{d\sec^{-1}(x)}{dx} = \frac{1}{\sqrt{1+x^2}\cdot(x)}$$

Sum, Product, Quotient and Chain Rules

$$\frac{d\,\text{constant}}{dx} = 0 \qquad \frac{d\,a\cdot f(x)}{dx} = a\frac{d\,f(x)}{dx} \qquad \frac{d\,f(x) \pm g(x)}{dx} = \frac{d\,f(x)}{dx} \pm \frac{d\,g(x)}{dx}$$

$$\frac{d\,f(x)\cdot g(x)}{dx} = \frac{df(x)}{dx}\cdot g(x) + f(x)\cdot\frac{dg(x)}{dx} = (f(x)\cdot g(x))' = f'(x)\cdot g(x) + f(x)\cdot g'(x)$$

$$\frac{d\,f(x)/g(x)}{dx} = \frac{\frac{df(x)}{dx}\cdot g(x) - f(x)\cdot\frac{dg(x)}{dx}}{g(x)^2} = (f(x)\,/\,g(x))' = \frac{f'(x)\cdot g(x) - f(x)\cdot g'(x)}{g(x)^2}$$

$$\frac{d\,f\circ g(x)}{dx} = \frac{d\,f(g(x)}{dx} = \frac{d\,f(g(x))}{d\,g(x)}\cdot\frac{d\,g(x)}{dx} = f'(g(x))\cdot g'(x) \qquad \text{or}$$

function: g( ) ○ f( )

x → g(x) → f(g(x))

derivative: $g'(x)$ • $f'(g(x))$

Implicit Differentiation

To compute $y'(x,y)$ for $F(x,y) = G(x,y)$, differentiate both sides, solve for y'.

Equation of Tangent Line

For $y = f(x)$ at $(a, f(a))$: $y = f(a) + f'(a)(x - a)$

For $F(x,y) = G(x,y)$ with $F(a,b)=G(a,b)$: $y = b + y'(a,b)(x - a)$

Differentials $dy = f'(x)\,dx$

where dy & dx are infinitesimally small changes (manipulate as variables).

Analyzing Functions

I. From $y = f(x)$:

- **Domain** remove x where **f(x) DNE @ $\div 0$, $\sqrt{<0}$, $\log(\leq 0)$**
- **X-int @ $f(x) = 0$** **Y-int @ $f(0)$**
- **Vert.Asym. $x = \#$: @ $\div 0$ or $\log(0)$**
- **Hori.Asym. $y = \#$: if $f(x) \to \#$ as $x \to \infty$**
- **Symmetry** **Odd if $f(-x)=-f(x)$** **Even if $f(-x)=f(x)$**
- **Period** **$\sin(ax)$** & **$\cos(ax)$** have **period = $2\pi/a$.**

II. From $y' = f'(x)$: (slope of tangent)

- **Critical Pt (CP) @ $f'(x) = 0$ or DNE**
  - **Local Max $f'(x)$ changes from >0 to <0 or $f''(CP)<0$**
  - **Local Min $f'(x)$ changes from <0 to >0 or $f''(CP)>0$**
- **$f'(x)>0$:$f(x)\uparrow$ increasing** **$f'(x)<0$:$f(x)\downarrow$ decreasing**

III. From $y'' = f''(x)$: (curvature)

- **DiCritical Pt @$f''(x) = 0$ or DNE**
  - **Inflection $f''(x)$ changes sign**
- **$f''(x)<0$:$f'(x)\downarrow$:ConcaveDown** **$f''(x)>0$:$f'(x)\uparrow$:ConcaveUp**

To find location of optima (maxima or minima):

1. **Model as a function, f(x), on some interval [a,b];**
2. **Find critical points, c, and compute f(c) all such c in [a,b];**
3. **Compare f(a), f(b) and all f(c) to find maximum or minimum.**

Anti-Derivatives and Integration

An anti-derivative of y = f(x) is a function y = F(x) , with F′(x) = f(x) .
(It is unique up to an arbitrary constant.)

Anti-derivative Value Problem: Find the anti-derivation F(x) for a function f(x) subject to the (initial) conditions: F(a) = b.

Indefinite Integral of f(x) is: $F(x) = \int f(t)dt$, $where\ F'(x) = f(x).$

(F(x) is a family of functions differing by an arbitrary constant.)

Fundamental Theorem of Algebra: For y = f(x) continuous on [a,b]:

(i) The (signed) area under the curve of y = f(x) from x=a to x=b is:

$$\int_a^b f(x)dx = [F(x)]_a^b = F(b) - F(a) \; ; \; where\ F'(x) = f(x).$$

(ii) $If\ F(x) = \int_a^x f(t)dt$, $then\ F'(x) = f(x).$

Elementary Functions:

$$\int x^a\, dx = \frac{x^{a+1}}{a+1} \, , \; for\ a \neq -1 \, ; \quad or = ln(x) \, , for\ a = -1;$$

$$\int e^{ax}\, dx = \frac{1}{a} e^{ax} \; ; \qquad \int ln(x)dx = xln(x) - x \, ;$$

$$\int sin(ax)dx = \frac{-1}{a} cos(ax) \, ; \qquad \int cos(ax)dx = \frac{1}{a} sin(ax) \, ;$$

$$\int sec^2(ax)dx = \frac{1}{a} tan(ax) \, ; \qquad \int tan(x)dx = ln\big(sec(x)\big) \, ;$$

$$\int sec(ax)tan(ax)dx = \frac{1}{a} sec(ax) \, ;$$

$$\int sec(x)dx = ln\big(sec(x) + tan(x)\big);$$

$$\int \frac{1}{1+x^2}\, dx = arctan(x) \, ; \qquad \int \frac{1}{\sqrt{1-x^2}}\, dx = arcsin(x)$$

Techniques of Integration

Linear: $\int (\, af(x) + bg(x)\,)dx \;=\; a\int f(x)dx \,+\, b\int g(x)dx$;

Substitution: $\int f(g(x))g'(x)dx \;=\; \int f(u)du$; $where\; u = g(x)$;

Parts: $\int u(x)dv(x) \;=\; u(x)v(x) - \int v(x)du(x)$;

Rational Functions: $\int {}^{P(x)}\!/_{D(x)}\, dx \;=\; \int Q(x)dx \,+\, \int {}^{R(x)}\!/_{D(x)}\, dx$;

where P(x) = D(x)·Q(x) + R(x) after polynomial division,

with Q(x) the quotient and R(x) the remainder

satisfying degree of R(x) < degree of D(x).

Partial Fractions: $\int {}^{R(x)}\!/_{D(x)}\, dx \;=\; \int \frac{c_1}{x - a_1} dx \;+ \cdots + \int \frac{c_n}{x - a_n}\, dx$;

where the $x - a_k$ are the factors of D(x)

and the c_k are determined by cover-up method.

Trig Tricks are too complicated to summarize and not that useful.

Improper Integration

$$\int_a^\infty f(x)dx = \lim_{b\to\infty} \int_a^b f(x)dx; \quad \int_{-\infty}^\infty f(x)dx \;=\; \int_{-\infty}^0 f(x)dx + \int_0^\infty f(x)dx.$$

Taylor Series and Analytic Functions

$f(x) = \sum_{n=0}^{\infty} c_n (x - a)^n$; where $c_n = f^{(n)}(a)/n!$, $for \mid x - a \mid < R$,

where R = radius of convergence = $\lim_{n\to\infty} \mid \frac{c_n}{c_{n+1}} \mid$.

Some common ones:

$$1/(1 - x) = \sum_{n=0}^{\infty} x^n \text{ , } for \mid x \mid < 1.$$

$$1/(1 + x^2) = \sum_{n=0}^{\infty} (-1)^n x^{2n} \text{ , } for \mid x \mid < 1.$$

$$arctan(x) = \sum_{n=0}^{\infty} (-1)^n x^{2n+1}/(2n + 1) \text{ , } for \mid x \mid < 1.$$

$$ln(x) = \sum_{n=0}^{\infty} (-1)^n (x - 1)^n/n \text{ , } for \mid x - 1 \mid < 1.$$

$$e^x = \sum_{n=0}^{\infty} x^n/n! \text{ , } for \mid x \mid < \infty.$$

$$cos(x) = \sum_{n=0}^{\infty} (-1)^n x^{2n}/(2n)! \text{ , } for \mid x \mid < \infty.$$

$$sin(x) = \sum_{n=0}^{\infty} (-1)^n x^{2n+1}/(2n + 1)! \text{ , } for \mid x \mid < \infty.$$

$$e^{a+bi} = e^a \left(cos(b) + isin(b)\right);\ DeMoive'sFormula.$$