Tensions over Tension

Abstract: This dramatization is based upon an actual Calculus II class. As is not uncommon, the students took unplanned control of a class and I merely moderated. The names of both the innocent and the guilty have been disguised in homage to Imra Lakatos's classic "Proofs and Refutations".

Professor Pi: (Entering the classroom to find the students engaged in a vehement discussion) What's going on here?

Beta: Alpha's being ridiculous. Professor Psi says tension is a force.

Professor Pi: Is this physics homework?

Gamma: Yes. A string is attached to a ball. You swing it in a circle, creating tension in the string. What direction is the tension?

Beta: It's circular motion. The acceleration is directed toward the center. The force, which is the tension in the string, must also be toward the center, toward your hand holding the string. It's basic physics. Alpha says it doesn't make sense.

Alpha: It's not that simple. Is the tension the same as the force? The tension exists throughout the string, not just at the end.

Beta: The tension is the force. At the ball's end, the force is towards your hand.

Alpha: But at your hand's end, the other end, the force must be towards the ball, the opposite direction. Where does it switch directions? Is the tension zero in the center? It doesn't make sense.

Gamma: Why can't the tension be zero in the center?

Delta: Do you play guitar? At the bridge end of a guitar string, the tension is toward the neck. At the neck end, the tension is toward the bridge. But it's not zero in the center. You can feel the tension in the center. When you pull the string out, the force is toward both the neck and the bridge.

Beta: Et tu, Delta? Are you siding with Alpha?

Delta: No. I just think Alpha has a point. It's not that simple.

Gamma: Is the force really in both directions?

Kappa: I play guitar. The force is like perpendicular to the direction you pull - like back to the resting position.

Alpha: The two opposing forces are now at an angle. The net force is the vector sum, which cancels everything except the perpendicular component. When the string is straight, it cancels completely and there is no net force. In the middle, the tension is a force with two directions that cancel. At the ends, there is only one direction. Is tension a vector with two directions, except at the ends?

Beta: Names, schmames. The physics is simple: the force at the ball's end is towards the hand. Professor Psi says so. Period.

Professor Pi: I'm not going to contradict Professor Psi. The tension creates a force at each end and that force is a vector. What happens in the middle does not affect the physics of the ball and the hand.

Delta: So what does happen in the middle? Is tension a vector or a scalar?

Professor Pi: It's actually kind of complicated. What do we mean by tension?

Gamma: The stress in the string.

Professor Pi: Stress - very good. Fundamentally, stress is about distortions to the electric field among the component atoms, but that's too difficult to analyze. Macroscopically, stress is … . What do you all know about water pressure?

Omega: I scuba dive. Water pressure varies with depth - every 33 feet the water pressure increases by one atmosphere. At a given depth, it's the same in all directions. It does not matter whether your face mask is pointing up or down, you feel the pressure on your face.

Alpha: So it does not have ***a single*** direction, either, it has every direction. Is it a scalar? But it creates a force and that's a vector?

Professor Pi: As it happens, I had planned to discuss water pressure today. The units of water pressure are force per area. Water pressure is created by gravity, by the weight of the water above a given depth. Consider $\mathbf{1\ m^3}$ of water. At the bottom, it sits on $\mathbf{1\ m^2}$ of area at a depth of **1 m .** What is the pressure? Who knows the density of water?

Beta: That's easy. A $\mathbf{cm^3}$ of water weighs, I mean, has a mass of one gram - that's the original definition of a gram. So $\mathbf{1\ m^3}$ has a mass of **1000 kg**.

Professor Pi: So, the force is **mg = 9800 N** . The pressure is $\mathbf{9800\ N/m^2}$. Now, **33 ft** is about **10 m** and a **pound** of force is about **4.4 N** . Under **33 ft** of water, the pressure is about $\mathbf{14.6\ lbs/in^2}$ or one atmosphere, as Omega said.

Gamma: But gravity is a downward force. Weight is downward. Why is the water pressure in all directions and not just down?

Professor Pi: Water is a fluid. The atoms move freely in all directions, so the downward force is transmitted as an equal stress in all directions. In a fluid, unequal stress is evened out. In a solid, unequal stress remains.

Alpha: If water pressure is stress acting equally in all directions, when does water pressure become an actual force acting in a specific direction?

Professor Pi: Omega can answer that for us. When you dive, Omega, and feel the force on your face mask, what is your mask relative to the water and your face?

Omega: I don't understand. I mean, it separates your face from the water. It's … like a boundary.

Professor Pi: Excellent. It's a boundary to the water. A boundary is a surface which determines a vector in the direction perpendicular to the surface - the normal vector. This is rapidly getting beyond the mathematics you know.

Delta: That's OK. We can understand it.

Professor Pi: OK. Water pressure is a function-like gizmo with a magnitude (force per area varying with depth) but no direction (it's in all directions). Input a vector as input to this gizmo (add a boundary), and it outputs a new vector (the force acting upon the surface in a direction normal to it). This sort of mathematical gizmo - a function with a varying magnitude which takes a vector as input and yields a vector as output - is called a tensor.

Delta: When do we learn tensors?

Professor Pi: You'll need to learn linear algebra and Calculus III first, which are your next two math courses. After that, most colleges do not have courses in "Tensor Analysis". You learn about tensors in Differential Geometry or in physics and engineering courses where tensors are needed to mathematically model stress, which includes General Relativity, a theory of gravity as stress in space-time.

Beta: But what does all this have to do with tension in a string?

Alpha: I think I see. The tension is a stress throughout the string. It has a magnitude with the units of force and no direction. At the boundary - the two ends - you input a vector - the inward direction - and the tension outputs a force directed inward. It's a tensor.

Delta: And when you pull the string outward in the middle, you add a double boundary - it bounds two regions. So you get two vectors, one into each region, and they net to the perpendicular force.

Gamma: Why can't you just think of it as inputting a vector perpendicular to the string and it outputs another vector perpendicular to the string?

Alpha: Because the string is basically one dimensional and the tension can only act in directions parallel to the string. The boundary is a point, which is zero-dimensional, I guess.

Gamma: So surface tension would be a two-dimensional stress tensor?

Omega: Or stress in a two-dimensional sailboat hull or airplane wing?

Professor Pi: Yes. Very good, all of you. You were all were very astute to realize that something complicated was going on with tension. It is complicated. Professor Psi is correct that the easiest way to deal with it is to consider tension as a force with a direction at the ends. Just as your calculus textbook considers water pressure to be a force at a boundary. But in reality, both tension and water pressure are tensors - stressful topics.

Delta: Oh, that's worse than your usual puns, Professor Pi.

Professor Pi: And now it's back to the stressful topic of calculus. What is the total force acting on the Hoover Dam?