<u>Math191 - Practice Test on Chapters 3</u>

1. Find the derivative of each of the following.

a) $f(x) = x^3 \cdot e^{2x}$ **(Product Rule)**

$$f'(x) = (x^3)'e^{2x} + x^3 \cdot (e^{2x})' = 3x^2 \cdot e^{2x} + x^3 \cdot 2e^{2x} = x^2 \cdot e^{2x}(3+2x)$$

$$\frac{d\, x^3 \cdot e^{2x}}{dx} = \frac{d\, x^3}{dx} e^{2x} + x^3 \frac{de^{2x}}{dx} = 3x^2 \cdot e^{2x} + x^3 \cdot 2e^{2x} = x^2 \cdot e^{2x}(3+2x)$$

b) $f(x) = \ln(\sqrt{x^2 + 1}$ $\quad : x \rightarrow x^2 + 1 \xrightarrow{(\ )^{1/2}} \sqrt{x^2 + 1} \xrightarrow{\ln()} \ln\sqrt{x^2 + 1}$

$f'(x)$: $\quad 2x \cdot (1/2)(x^2 + 1)^{-1/2} \cdot ((x^2 + 1)^{1/2})^{-1}$

$$f'(x) = x(x^2 + 1)^{-1} = \frac{x}{(x^2 + 1)} \qquad .$$

$$\frac{d \ln(x^2 + 1)^{1/2}}{dx} = \frac{d \ln(x^2 + 1)^{1/2}}{d(x^2 + 1)^{1/2}} \cdot \frac{d(x^2 + 1)^{1/2}}{d(x^2 + 1)} \cdot \frac{d(x^2 + 1)}{dx} = (x^2 + 1)^{-1} \cdot x$$

c) $f(x) = \sin^2(x^2)$ $\quad : x \rightarrow x^2 \xrightarrow{\sin()} \sin(x^2) \xrightarrow{()^2} \sin(x^2)^2$

$f'(x)$: $\quad 2x \cdot \cos(x^2) \cdot 2\sin(x^2)$

$$f'(x) = 2x \cdot \cos(x^2) \cdot 2\sin(x^2) = 4x \cdot \cos(x^2) \cdot \sin(x^2) = 2x\sin(2x^2)$$

$$\frac{d \sin^2(x^2)}{dx} = \frac{d \sin^2(x^2)}{d \sin(x^2)} \cdot \frac{d\sin(x^2)}{d(x^2)} \cdot \frac{ds(x^2)}{dx} = 2\sin(x^2) \cdot \cos(x^2) \cdot 2x$$

d) $f(x) = x^2\ln(x^2)$ **Product Rule &** $: x \to x^2 \to \ln(x^2)$

$2x \quad \cdot (x^2)^{-1} = 2x^{-1}$

$$f'(x) = (x^2)'\ln(x^2) + x^2(\ln(x^2))' = 2x\cdot\ln(x^2) + x^2\cdot 2x^{-1} = 2x(\ln(x^2) + 1)$$

e) $f(x) = \sec(x / (1+x^2))$; this is

$= \sec \circ g(x)$; where $g(x) = \dfrac{x}{1 + x^2}$ **Quotient Rule.**

$$g'(x) = \frac{(1+x^2)(x)' - x(1+x^2)'}{(1+x^2)^2} = \frac{(1+x^2) - x(2x)}{(1+x^2)^2} = \frac{1 - x^2}{(1+x^2)^2}$$

$$f'(x) = \sec'(g(x))\cdot g'(x) = \frac{\sec(g(x))\cdot\tan(g(x))\cdot(1-x^2)}{(1+x^2)^2}$$

f) $f(x) = \dfrac{\tan^{-1}(x)}{1 + x^2}$ **Quotient Rule**

$$f'(x) = \frac{(\tan^{-1}(x))'\cdot(1+x^2) - \tan^{-1}(x)\cdot(1+x^2)'}{(1 + x^2)^2}$$

$$= \frac{(1+x^2)^{-1}(1+x^2) - \tan^{-1}(x)\cdot(2x)}{(1 + x^2)^2} = \frac{1 - 2x\cdot\tan^{-1}(x)}{(1 + x^2)^2}$$

g) $f(x) = \sqrt{x + \sqrt{x}}$ $: x \longrightarrow x + x^{1/2} \xrightarrow{()^{1/2}} (x + x^{1/2})^{1/2}$

$$f'(x) = (1 + \tfrac{1}{2}x^{-1/2}) \bullet \tfrac{1}{2}(x + x^{1/2})^{-1/2}$$

$$= \frac{2\sqrt{x} + 1}{4\sqrt{x}\cdot\sqrt{x + \sqrt{x}}}$$

$$\frac{d\,(x + x^{1/2})^{1/2}}{dx} = \frac{d\,(x + x^{1/2})^{1/2}}{d(x + x^{1/2})}\,\frac{d(x + x^{1/2})}{dx} = \tfrac{1}{2}(x+x^{1/2})^{-1/2}(1+\tfrac{1}{2}x^{-1/2})$$

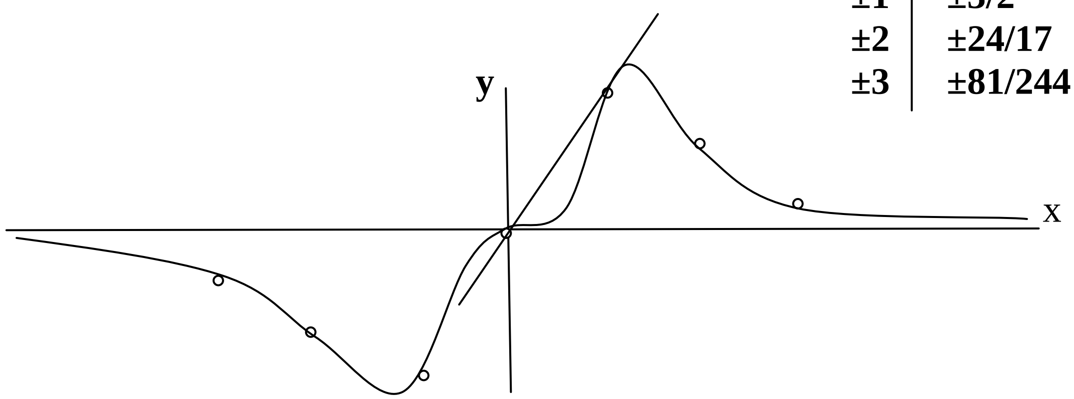
h) $f(x) = e^{\sin(2x)}$: $x \xrightarrow{\sin()} \sin(2x) \xrightarrow{e^{()}} e^{\sin(2x)}$

$f'(x) = 2\cos(2x) \bullet e^{\sin(2x)}$

$$\frac{d\, e^{\sin(2x)}}{dx} = \frac{d\ e^{\sin(2x)}}{d\sin(2x)}\,\frac{d\sin(2x)}{d2x}\,\frac{d\,2x}{dx} = e^{\sin(2x)}\cdot\cos(2x)\cdot 2$$

2.<u>Graph</u> the function: $f(x) = \dfrac{3x^3}{x^4+1}$

x	f(x)
0	0
±1	±3/2
±2	±24/17
±3	±81/244



$$f'(x) = \frac{(x^4+1)9x^2 - 3x^3 4x^3}{(x^4+1)^2} = \frac{-3x^6 + 9x^2}{(x^4+1)^2} = \frac{3x^2(3-x^4)}{(x^4+1)^2}$$

Domain = {all x}; **Range =** $\left\{ \dfrac{-3^{7/4}}{4} \le y \le \dfrac{3^{7/4}}{4} \right\}$ (hard – not on test)

X-int = (0,0) = Y-int.

No Vertical Asymptotes (no logs nor divide-by-0) .

$$\lim_{x\to\infty} f(x) = \lim \frac{3x^3}{x^4+1} \approx \frac{3}{x} = 0$$ **; so y = 0 Horizontal Asymptote.**

Tangent at **x=1,** $f(1) = {}^3/_2 = 1.5$, $f'(1) = {}^6/_4 = 1.5$: $y = 1.5 + 1.5(x-1)$ **;**
thus, tangent line is: $y = 1.5x$ **.**

3.Calculate the following limits:

a) $\displaystyle \lim_{x\to0} \frac{\sin(3x)}{5x}\cdot\frac{3}{3} = \frac{3}{5}\lim_{3x\to0}\frac{\sin(3x)}{3x} = \frac{3}{5}$ **(using sinx/x → 1)**

b) $\displaystyle \lim_{x\to\infty} \frac{-1}{5x} \le \lim_{x\to\infty}\frac{\sin(3x)}{5x} \le \lim_{x\to\infty}\frac{+1}{5x}$

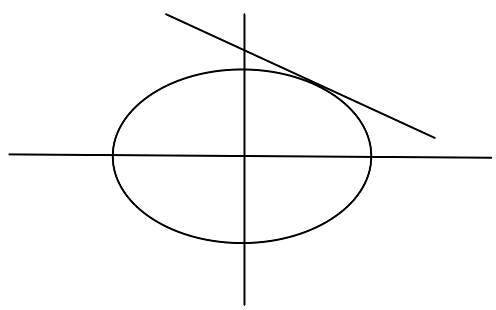
$-1/\infty = 0 \quad \le \quad ? \quad \le \quad 0 = 1/\infty$ **; so, ? = 0, too.**

c) $\displaystyle \lim_{x\to0}\frac{e^{3x}-1}{x}\cdot\frac{3}{3} = 3\lim_{3x\to0}\frac{e^{3x}-1}{3x} = 3$ **(using def'n of e)**

4.Consider the ellipse: $4x^2 + 9y^2 = 36$.
Find the derivative, y', and the equation of the tangent line at $(\sqrt{5}, {}^4/_3)$?

Check: $4(5) + 9({}^{16}/_9) = 20 + 16 = 36$, OK.

Differentiate both sides:



$8x + 18yy' = 0$

$y' = \dfrac{-8x}{18y} = \dfrac{-4x}{9y}$

$y'(\sqrt{5}, {}^4/_3) = \dfrac{-4\sqrt{5}}{9(4/3)} = \dfrac{-\sqrt{5}}{3}$

Tangent: $y = \dfrac{4}{3} + \dfrac{-\sqrt{5}}{3}(x - \sqrt{5}) = \dfrac{-x\sqrt{5}}{3} + 3$

The graph looks about right, too.

5.A mass on a spring moves left and right with displacement **s(t)** from the equilibrium point according to the function:

$s(t) = 2\cos(3t)$ Period: **$3t: 0 \rightarrow 2\pi$,** or **$t: 0 \rightarrow {}^{2\pi}/_{3}$ secs**

$v(t) = s'(t) = -6\sin(3t)$

$v(t) < 0$ for $0 < t < {}^{\pi}/_{3}$
$v(t) > 0$ for ${}^{\pi}/_{3} < t < {}^{2\pi}/_{3}$

$a(t) = v'(t) = -18\cos(3t)$

$a(t) < 0$ for $0 < t < {}^{\pi}/_{6}$
$a(t) > 0$ for ${}^{\pi}/_{6} < t < {}^{\pi}/_{2}$
$a(t) < 0$ for ${}^{3\pi}/_{6} < t < {}^{2\pi}/_{3}$

For speeding up (graph speed = |v(t) |): $0 < t < {}^{\pi}/_{6}$ & ${}^{\pi}/_{3} < t < {}^{\pi}/_{2}$

6.a) The radioactive isotope of cobalt, **^{60}Co** , has a half-life of 5.3 years. What percent of a sample of pure **^{60}Co** remains after 25 years?

Let **A(t)** be amount after **t** years.

$A(t) = A(0)2^{-t/H} = 100\cdot2^{-t/5.3}$; $A(25) = 100\cdot2^{-25/5.3} = 3.80$: 3.8%

or

$A(t) = A(0)e^{-t\ln2/H} = 100\cdot e^{-t\ln2/5.3}$; $A(25) = 100\cdot e^{-25\ln2/5.3} = 3.80$

Check: Divide 5.3 into 25, and it is halved 4.7 times – about 4.

Altlernate 6.a) Jackelopes look like jack rabbits with antlers. They are shy and seldom seen. They breed like rabbits, but fight like antelopes. A population of jackalopes doubles every **6** years. If **20** jackelopes, half males and females, are introduced into a unsuspecting New Zealand wilderness, how long before the population reaches **100,000**?

Let **P(t)** be population after **t** years.

$P(t) = P(0)2^{t/d} = 20\cdot2^{t/6}$ **=** which equals 100,000 when **$2^{t/6} = 5000$,**
so when **$t = 6\log_2(5000) = 73.7$ years.**

6.b) A warm human body, liver temperature **95°F**, if left outside overnight for **12** hours at **50°F**, has a liver temperature of **55°F**. A murder victim is discovered outside at **7AM**. The evidence supports the assumption the body has lain there overnight at (about) **50°F**. The liver temperature was found to be **70°F**. At what time was the person killed? (Assume a body temperature of **99°F.**)

Let **T(t)** = temperature after **t** hours; $\mathbf{T_s = 50°F}$ = surrounding temperature.

$$\mathbf{T'(t) = a(T(t) - T_s);\ \ so\ \ T(t) - T_s = (T(0) - T_s)e^{-at}}\ .$$

Using test warm body data:

$$\mathbf{T(12) = 55 = 50 + (95 - 50)e^{-a12}\ ;\ \ ln(5/45) = -12a\ ;\ a=ln(9)/12=0.183}$$

Using murder victim data:

$$\mathbf{T(t) = 70 = 50 + (99 - 50)e^{-0.183t}\ ;\ \ ln(20/49) = -0.183t\ ;}$$

$$\mathbf{t = ln(49/20)/0.183 = 4.9\ hours = 4\ hours,\ 54\ minutes.}$$

Killed at: 2:06AM.

6.c) For x units manufactured, let the (total) cost be **C(x),** average cost (per unit) is **A(x)=C(x)/x,** and marginal cost (per unit) is **C′(x)**.
What is **A′(x)** ?

By quotient rule: $$\mathbf{A'(x) = \frac{x \cdot C'(x)\ -\ 1 \cdot C(x)}{x^2}}$$

So, **A′(x) = 0 when A(x) = C(x)/x = C′(x) .** So, the average cost per unit has slope zero when it equals the marginal post per unit. Notice **A(x)** is slope of **secant** connecting **(0,C(0)) to (x,C(x))**, while **C'(x)** is slope of **tangent** connecting **at (x,C(x)).** This gives a graphical solution.