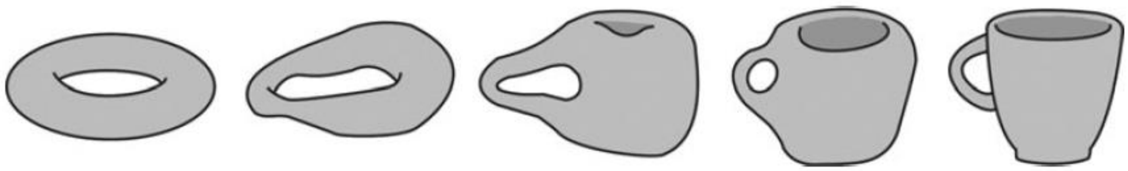
The Shape of Things

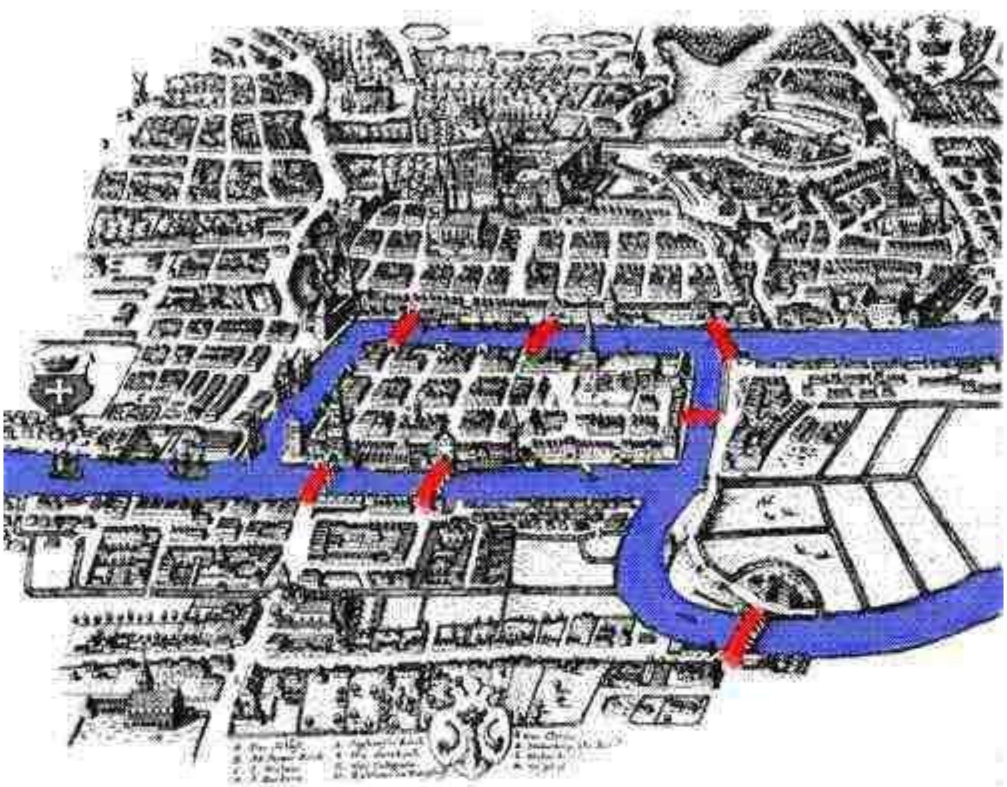
An Introduction to Topology

Is a donut the same shape as a coffee cup? Think of a lump of clay. Just reshape it without tearing. One can reshape a donut into a coffee cup.

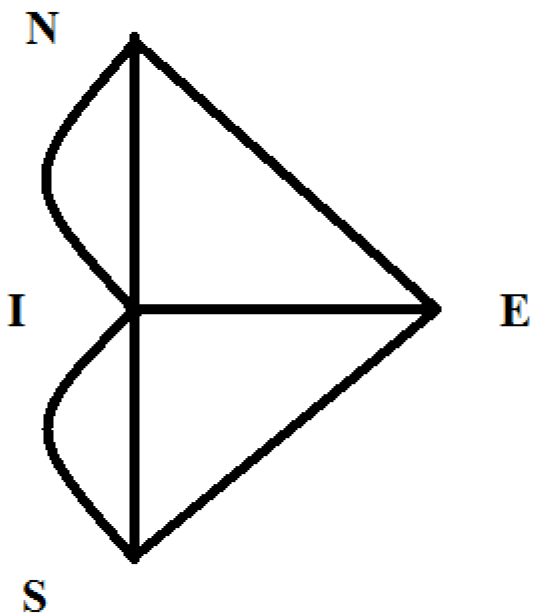


But to turn a ball or bowl into a donut, one must punch a hole. One must tear it.

Topology is about **shape** and **continuity**. It began with **Euler** and the bridges of **Konigsberg**. Could one walk through the city and traverse each bridge precisely once?



Euler proved the answer was “No”. He abstracted the complexity of the map into a simpler situation. In doing so, he invented **graph theory.**



The basic concepts of graph theory are:

graph is a collection of **vertices** all connected by **edges**;
path is a sequence of adjacent vertices with no edges repeated;
circuit is a path that ends where it starts;
degree of a vertex = number of edge endings at the vertex.

The Konigsberg problem asks whether a **Global (Euler) Path or Circuit**, one which traverses every edge, exists. Euler realized that **Odd (degree) vertices** pose a problem for global paths.

Theorem: A graph has a global circuit if and only if there are no odd vertices.

The Theorem is about the shape of a graph. If there are no odd vertices, the graph is equivalent to a circle immersed in the plane, overlaid upon itself so that there are self-intersections. The number of odd degree vertices divided by two characterizes the underlying shape of the graph. It is a **topological invariant**. If zero, the graph is basically a circle. If one, the graph is basically just a single line segment. If two, the graph is composed of two separate line segments, but not any fewer. Etc. The result is typical of many topological theorems in that it mixes **local** and **global** ideas.

Euler continued to study graph theory:

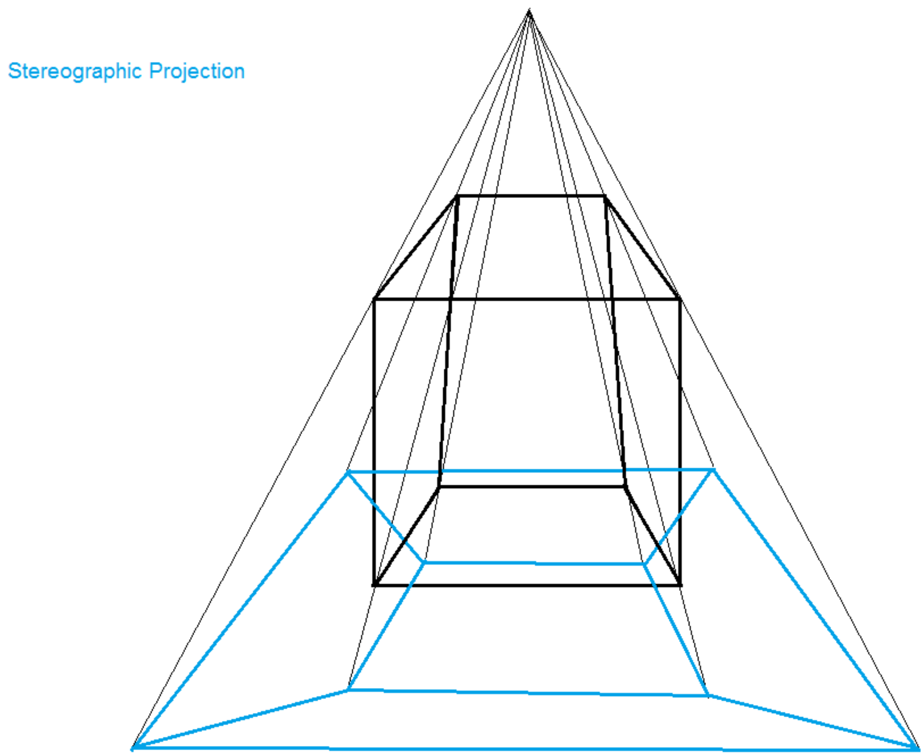
> **simple graph** is one without loops or multiple edges;
> **planar graph** is a simple graph drawable without crossed edges.

A planar graph divides the plane into **regions** bounded by the edges and, also, one unbounded region. Euler proved:

Theorem: For any simple planar graph:

> **# of vertices – # of edges + # of regions = 2**.

Euler noticed there is a relation between planar graphs and convex polyhedrons.

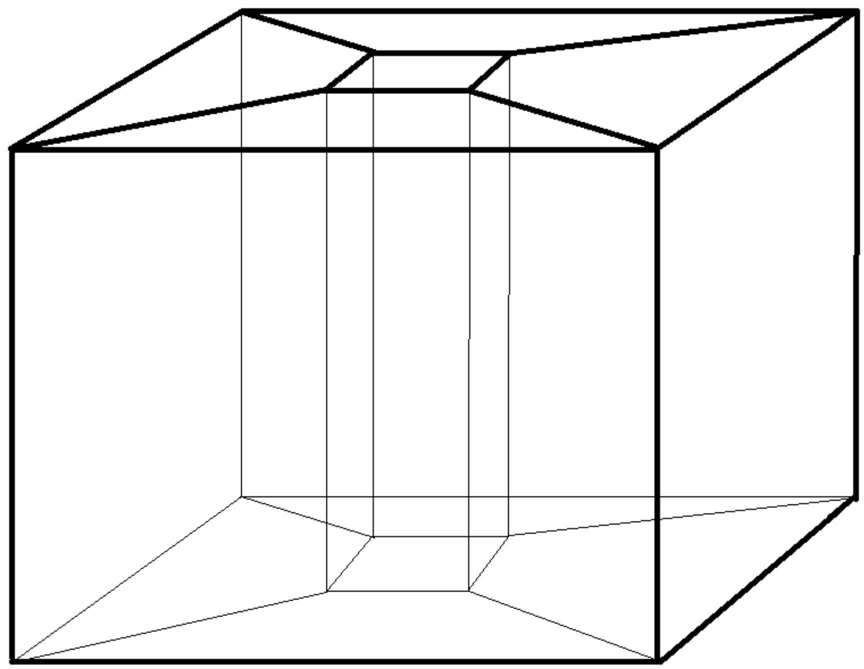


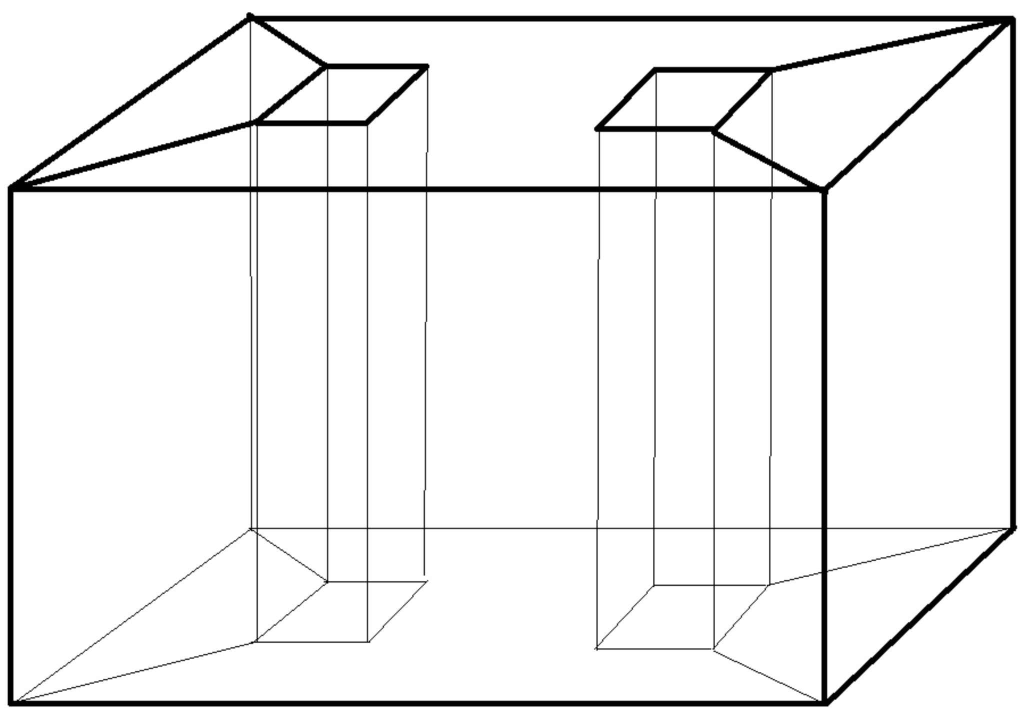
The regions, even the unbounded region, become faces in a polyhedron, and the result becomes:

# of vertices – # of edges + # of faces = 2.

The same result is true for all the Platonic Solids: tetrahedron, hexahedron, octahedron, doadecahedron and icosahedron. These and other convex polyhedron are basically Bizarro World spheres. Using stereographic projection, any planar graph becomes a convex polyhedron drawn as a graph on the surface of a sphere, which defines a “polyhedronization” of the sphere.

What about non-convex polyhedrons with holes? Donut? Bear Claw?



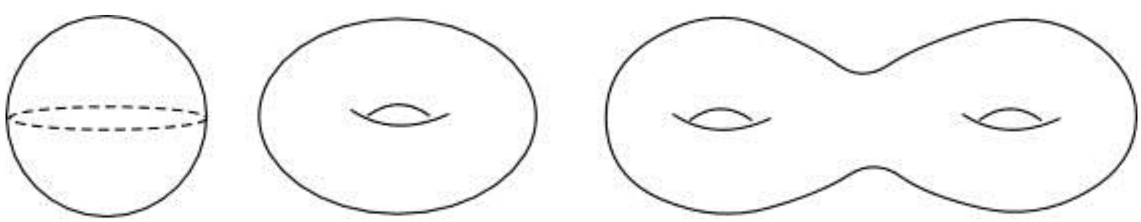


	# of vertices	–	# of edges	+	# of faces		
Donut:	**16**	**–**	**32**	**+**	**16**	**=**	**0**
Bear Claw:	**24**	**–**	**44**	**+**	**18**	**=**	**-2**

Notice that these totals are unchanged under subdivision (trianglization). For any **"polyhedronization"** of a given surface, the sum is the same:

# of vertices – # of edges + # of faces = χ , the Euler Characteristic.

This number characterizes the essential shape of a surface. It is a **topological invariant.** The surface of a ball is the same shape as a bowl and has invariant **2**. The surface of a donut is the same shape as a cup and has invariant **0**. The surface of a bear claw has invariant **-2**. A three-holed "bear claw" has **-4**.



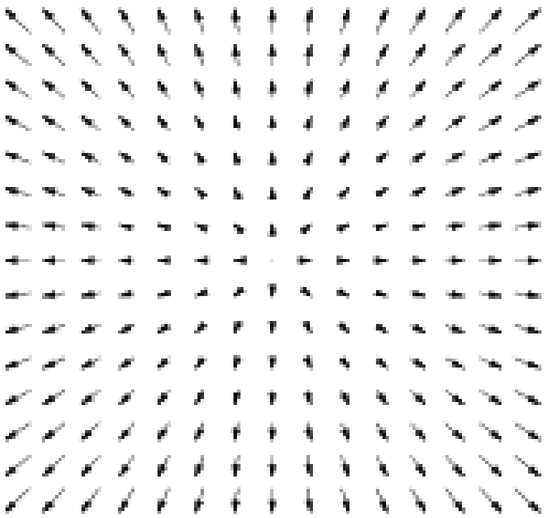
All of the above is based upon discrete mathematics – a finite number of vertices, edges and faces. Surfaces are a continuum. Continuity is a calculus concept. Is the Euler Characteristic related to any ideas in calculus?

Calculus begins with the idea of a tangent line: a smooth curve has a tangent line and everywhere looks locally like a piece of a line. Similarly, a smooth surface everywhere looks locally like a 2-dimensional plane. Each point on the surface has a **tangent plane**, usually defined as the space of all vectors tangent to the surface.

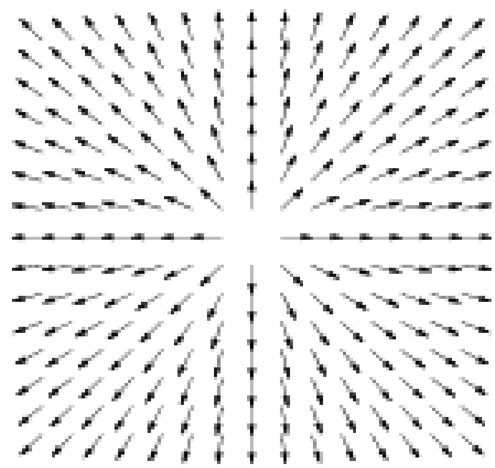
A **vector** is often pictured as an arrow with **length** and **direction**. In the plane, every point may be used as an origin for vectors. To add vectors, one parallel transports it between different origins. This **parallel transport** does not depend upon the piece-wise linear path. This is not true on spheres, with lines replaced by great circles. Transport a vector from the equator, up to the north pole, back down to the equator, and then around the equator back to the start. It's changed directions. A sphere is curved.

Consider a **vector field**, the assignment of a vector to each point of some subset of the plane. Basically, it is a mapping: $\mathbf{F(x,y) = (f_1(x,y), f_2(x,y))}$.

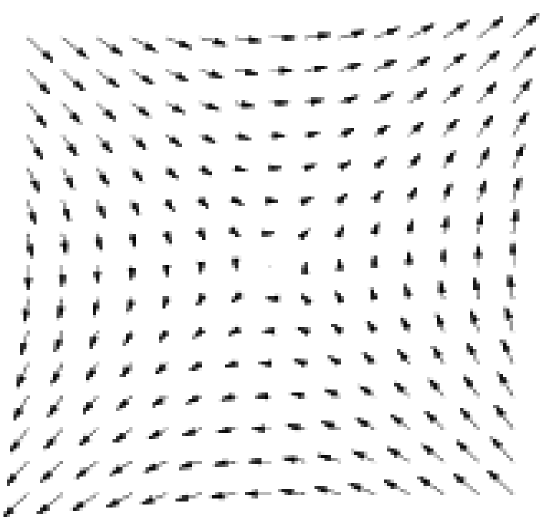
$\{x, y\}$



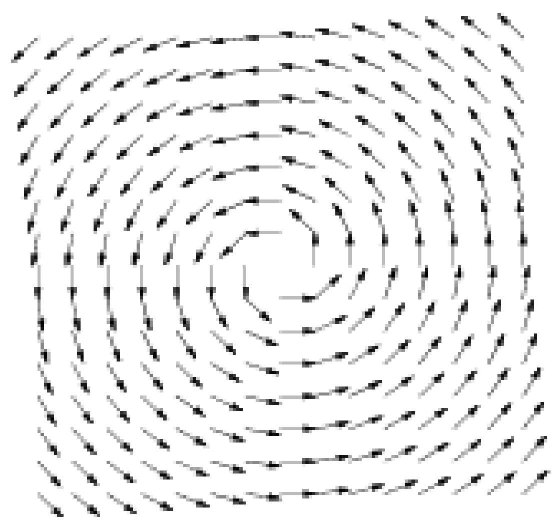
$\left\{ \frac{x}{\sqrt{x^2+y^2}}, \frac{y}{\sqrt{x^2+y^2}} \right\}$



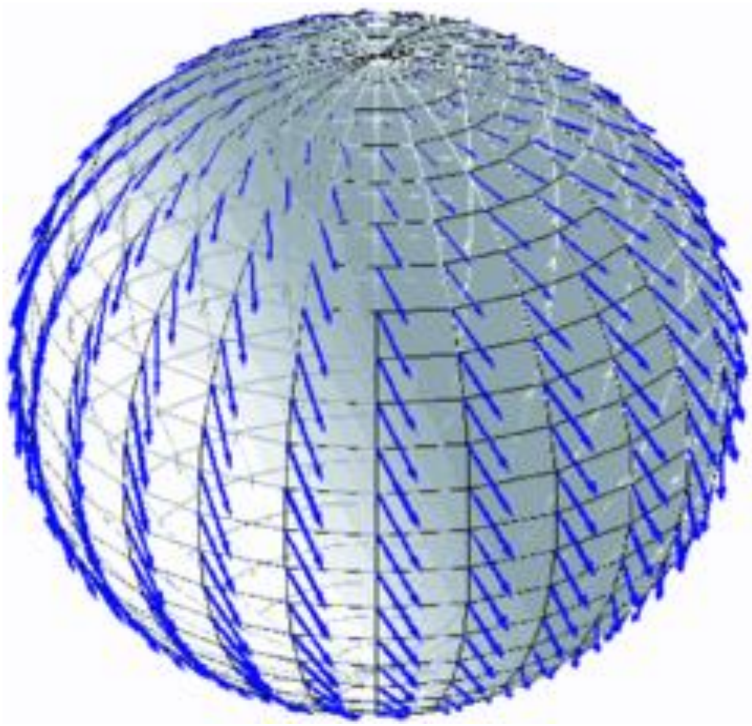
$\{y, x\}$



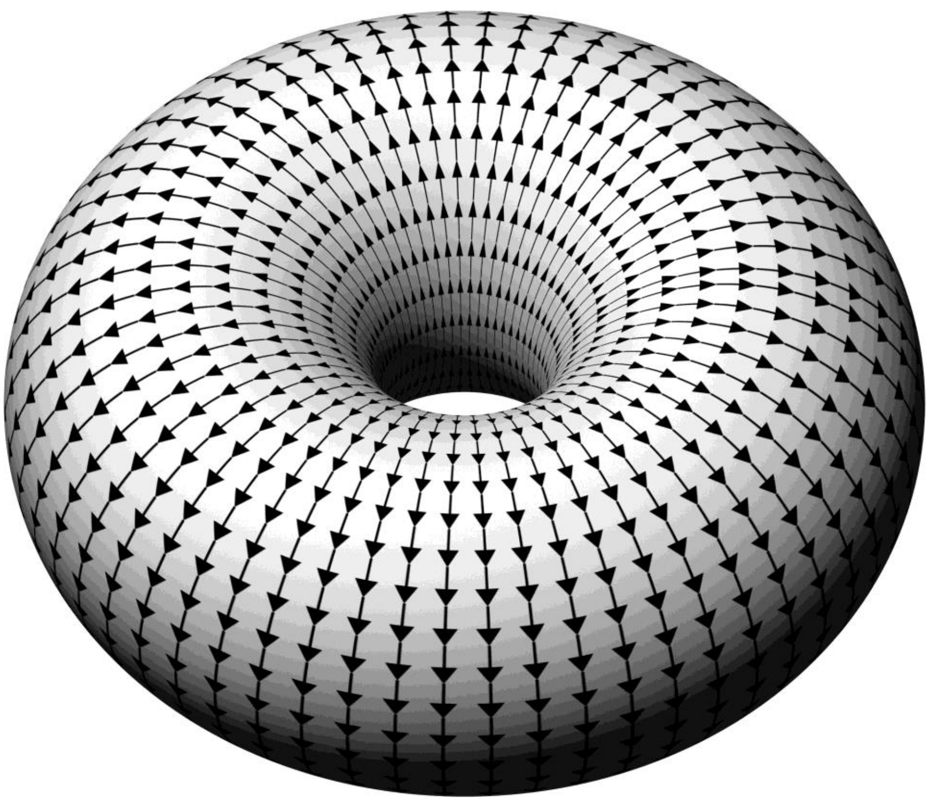
$\{-y, x\}$



Points where the vector field is zero are complicated. The vectors may swirl around. The number of times around is called the **index of the zero**. In the above, the indices are **+1, NA, -1,** and **+1.** In terms of complex numbers, $\mathbf{z = x + iy}$, $\mathbf{f(z) = z^n}$ defines a field of index **n** . Zeroes of vector fields on surfaces similarly have an index. Notice that vector fields on a sphere always have zeroes.



But there are vector fields on the torus which do not have zeroes.



Poincaré proved that for any vector field with a finite number of zeroes defined on a smooth surface:

$$\Sigma \text{ indexes } = \chi \text{, the Euler Characteristic.}$$

This is the mathematical basis for the famous **“Hairy Sphere Theorem”** which asserts that the hair cannot be combed flat over the whole sphere.

This result, properly generalized, is even true in higher dimensions. A 3-dimensional surface is called a hyper-surface or 3-manifold. One can picture a 3-sphere by adding a point at infinity to 3-space, just as one obtains a 2-sphere by adding a point at infinity to a 2-plane. And there is no reason to stop at three.