# Dimensional Transcendentalism

Abstract: This dramatization is based upon an actual Calculus II class. The names of both the innocent and the guilty have been disguised in homage to Imra Lakatos's classic "Proofs and Refutations".

Professor Pi: At the end of the last class, we solved the first order differential equation for free fall with air resistance. But we didn't have time to discuss it and there are some important lessons here. Let's recall the equation and the solution. Beta, would you write them on the board?

Beta: Let **v(t)** be the velocity in meters per second, with **t** in seconds. Let **g** be the constant acceleration of gravity. Let **κ** be the coefficient of air resistance. Newton's second law gives:

$$\mathbf{Force = m\cdot a\ =\ m\cdot v'(t)\ =\ -m\cdot g\ -\ \kappa\cdot v(t)\ ;\quad with\ \ v(0) = v_0\ .}$$

This is linear and the integration formula gives the solution:

$$\mathbf{v(t)\ =\ -m\cdot g/\kappa + (v_0 + m\cdot g/\kappa)\cdot exp(-\kappa\cdot t/m)\ .}$$

Professor Pi: Very good. I like your use of **"exp"** for the exponential function. What is the limit of **v(t)** as **t** $\rightarrow \infty$ ?

Beta: It's **-m·g/κ** , since the exponential approaches zero.

Professor Pi: What are the units of **m·g/κ**?

Beta: Well, **m** is kilograms, **g** is meters per second squared and **κ** is … it's not in my notes. You didn't tell us.

Gamma: Since **m·g/κ** is the limit of the velocity, it must have units of meters per second, so **κ** must have units of kilograms per second.

Professor Pi: Very good, Gamma. The units must balance is rule number one when doing physics. Let's set $\mathbf{m\cdot g/\kappa\ =\ v_\infty}$ and call it terminal velocity. We'll use the convention that all three, **g, m, κ and** $\mathbf{v_\infty}$ **,** are positive, and write the solution as:

$$\mathbf{v(t)\ =\ -v_\infty + (v_0 + v_\infty)\cdot exp(-\kappa\cdot t/m)\ .}$$

Beta: But if **κ** is kilograms per second, that's a rate of change and the mass isn't changing.

Alpha: If you look at the differential equation, it makes more since to say that **κ** is force per velocity, which is a rate of change: how the force of air resistance changes with velocity. The units simplify to kilograms per second.

Professor Pi: An excellent explanation, Alpha. What are the units of **κ·t/m**, the input to the exponential function?

Beta: It's kilograms per second times seconds divided by kilograms whichleaves nothing. Is that possible?

Professor Pi: A better word than "nothing" is "unitless" and in this case it is mandatory. Units are conserved. They must always balance and they are never destroyed. Only quantities with the same units may be added or subtracted. When multiplying or dividing quantities with units, the units must be multiplied or divided. The input to a transcendental function must be unitless.

Gamma: What's a transcendental function?

Professor Pi: It's a function that's not built up from the four basic operations of addition, subtraction, multiplication and division. They include exponentials, logarithms and trigonometric functions.

Gamma: What about taking roots?

Professor Pi: A good question. When taking a square root of a quantity with units, the units must be square units.

Delta: What about exponents? The formula for compound interest is:

$$\mathbf{B(t) = B(0)(1 + r)^{t}}$$ , where **t** is in years.

What happens to the years in the exponent?

Alpha: And **r** has units of "per year". What happens to that?

Professor Pi: Good questions. You need to use the full formula:

$$\mathbf{B(t) = B(0)(1 + r/m)^{tm}}$$ ; compounded **m** times per year for **t** years.

So both **r/m** and **mt** are unitless, while **B** has units of dollars.

Beta: What about trig functions? The input has units in radians?

Professor Pi: Another good question. What is the radian measure of an angle?

Delta: It's the length of the subtended arc divided by the length of the radius.

Professor Pi: Excellent. If you measure the lengths of the arc and the radius in centimeters, then the angle has units of centimeters divided by centimeters, which is unitless. Radians are really pure numbers and not true units.

Alpha: If you measure the angle in degrees, is that a unit?

Professor: Not really. It's a subtle point. I probably should use the word "dimensionless" rather than "unitless". Radians are indeed a unit of measurement, but they are not a unit for a true dimension. They are just pure numbers, like percent is a pure number. Degrees are just radians times $\mathbf{180/\pi}$ and also pure numbers. There are six fundamental dimensions. Who can name them?

Beta : Time, distance and mass are three.

Gamma: Electric charge?

Professor Pi: That's four. There's two more.

Beta: Energy?

Professor Pi: That's a derived dimension. Energy is mass times distance squared divided by time squared, as in Einstein's: $\mathbf{E = m \cdot c^2}$ , where **c** is the speed of light. There's two more dimensions.

Delta: I think they're temperature and brightness.

Professor Pi: Excellent. Time, distance, mass, charge, temperature and luminosity; with their respective units of seconds, meters, kilograms, Coulombs, °Kelvin and lumens.

Alpha: How about moles?

Professor Pi: That's another subtle point. Moles are just a count of molecules: about $\mathbf{6.02 \cdot 10^{23}}$ molecules. A count is just a number. Now, a second is defined as a certain count, too. It is some number of vibrations of a cesium atom in its ground state. But a second measures a property which exists apart from cesium atoms. Similarly, a meter was once defined as so many wave lengths of a certain reddish light emitted by a krypton atom. But a meter measures a property which exists apart from krypton atoms.

Gamma: Are any of the other units defined as counts?

Professor Pi: Officially, the answer is no. A coulomb could be defined as the charge of so many electrons, a gram as the mass of so many carbon atoms, and a lumen as so many photons of a certain wavelength. But notice that other particles besides electrons have charge, other atoms besides carbon have mass and other wavelengths have brightness. Dimensions measure fundamental properties of nature. The units by which we measure dimensions are, or could be, defined as counts, with the exception of temperature.

Delta: What is temperature?

Professor Pi: Temperature measures the "quality of heat", whereas energy, usually in calories, measures the "quantity of heat". A calorie is the quantity of heat required to raise the temperature of one gram of water 1°K. There are a hundred °K=°C between the freezing and boiling temperatures of water. It's a little circular. Temperature determines the direction heat flows, from warmer to cooler.

Alpha: Why is temperature so different?

Professor Pi: This is getting beyond mathematics and into physics and philosophy. While mass and charge are somewhat objective properties of nature, notice that the others are related to what Kant termed categories of human perception. Time measures duration and distance measures extension. Brightness is self-explanatory. Temperature measures "hotness" and it's not obvious what "hotness" is. In the Theory of Thermodynamics, temperature is related to the average kinetic energy of the atoms, but it does not have the units of energy.

Gamma: Who's Kant?

Alpha: He was a philosopher. Is there any deep reason why there are only six fundamental dimensions?

Professor Pi: Not that I know of. Perhaps this theory of everything the physicists are always talking about will answer that. In fact, there is at least one other dimension. The strong nuclear force involves another type of charge besides electric charge. But that's a quark of another color.

| | | |
|---|---|---|
| Note: | A second is: | **9,192,631,770** vibrations of $\mathbf{Cs_{133}}$. |
| | A meter was once: | **1,650,763.73** wavelengths of red light from $\mathbf{Kr_{86}}$. |
| | A gram could be: | $^{1}/_{12}$ the mass of $\mathbf{6.0221413 \cdot 10^{23}}$ $\mathbf{C_{12}}$ atoms. |
| | A coulomb could be: | the charge of $\mathbf{6.2415093 \cdot 10^{18}}$ electrons. |
| | A lumen is about: | $\mathbf{4.0 \cdot 10^{15}}$ photons of green light (**λ=550nm**). |