What is a Number, Really?

You know what a number is. Or do you?

If actually asked to define a number, you might pause – and say something like, “an amount” or “a quantity.” But that’s just definition by synonym – by word substitution. If asked for a better definition, you might pause, think a bit, shrug and say, "A number is ... like … one, two, three, … , you know, what you count with!” And actually, that's as close to a right answer as there is. (In the following, a familiarity with introductory algebra, including negatives, fractions and exponents, is assumed, but then reviewed from a logical-historical viewpoint.)

Bertrand Russell once defined the number two as that property “which all pairs have in common”. Whole numbers are properties that we assign to collections of objects. How many apples are in the bowl? You take them out, one by one, counting as you do, and when the bowl is empty, the number you stopped at is the answer.

All of mathematics begins with the concept of the natural numbers or counting numbers: one, two, three, four, five, six, … This concept actually has two parts: a naming process and a matching process.

When you learned to count, you first learned a naming process. You learned the name of the first one, which in English is not uncoincidentally "one"; and you learned the rule to name the next one: two follows one, three follows two, four follows three, etc. What comes after nineteen? Twenty, and repeat with twenty-one through twenty-nine. What comes after twenty-nine? Thirty, and repeat. What comes after ninety-nine? One hundred, and repeat. What comes after one hundred ninety-nine? Two hundred, and repeat. What comes after nine hundred ninety-nine? One thousand, and repeat. Etc.

But you learned more than just a naming rule; you also learned a matching process. To answer how many things are in some collection, you learned to touch the things one by one while saying the names of the numbers in their proper order, being careful not to touch any one thing more than once. When you run out of things, the name you are at is the answer to how many things there are.

If you take all this childhood learning and wrap it up in the fancy language of symbolic logic, you have the "Peano Axioms" for the natural numbers. But we’re not going to go there – that’s for an advanced course in abstract algebra or symbolic logic – we’re going to keep it simple and intuitive.

The whole Numbers are what you count with – that's good enough for now. But we need one more number if we are to count any collection. How many pink unicorns are in the room with you right now? None, right? The number for none is "zero". Simple as this seems, the concept of zero proved a very difficult one historically. Zero was not accepted as a genuine number until rather recently.

Written Representations of Numbers

How do we write whole numbers? Using words to write them, as we did in the above, fast becomes awkward. Writing "One thousand, three hundred twenty-five", while we may do so on checks, is a lot more effort than writing "1,325". This shorthand way of writing numbers is so familiar that we often forget how useful and convenient it is.

What is the English name for the number: **2,478,932,173,804,067,194,566,801?**

Few people can name it (**"two septillion, four hundred seventy-eight sextillion, nine hundred thirty-two quintillion, one hundred seventy-three quadrillion, eight hundred four trillion, sixty-seven billion, one hundred ninety-four million, five hundred sixty-six thousand, eight hundred one"** – unless you live in England, in which case the name is - **"two thousand four hundred seventy-eight trillions, nine hundred thirty-two million one hundred seventy-three thousand eight hundred four billions, sixty-seven thousand one hundred ninety-four millions, five hundred sixty-six thousands, eight hundred one"**).

But everyone can write the next number: **2,478,932,173,804,067,194,566,802**.

Naming numbers with words is difficult: naming them with our usual numeral notation is easy. This notation was invented by the Arabs (circa 600AD) and is called "Arabic numerals". In inventing it, the Arabs borrowed key ideas from the Hindus, who possibly borrowed the key idea of a placeholder zero from the even earlier Babylonians. Alternatively, the notation may have possibly been inspired by that Chinese invention – the first computer – the abacus – which was widely used by the Hindus.

Every known human culture invented numbers as part of that most important of all inventions – language. We know this because, in each major language group, the linguistic roots for the words for the small whole numbers are distinct; and hence, that original culture did NOT borrow these words from another culture. Each culture invented words for numbers as big as they needed. The aboriginal bushmen of Australia could count to ten - after that they said "many" and "many manies" – they didn't need to count higher as they did not have a lot of "stuff". The ancient Egyptians, with their millions of acres of fertile soil renewed annually by the flooding of the Nile, had lots of "stuff", and thus they invented words to count to the thousands and beyond.

Whenever written languages were invented (it occurred independently five or so times – China, India, Iraq, Egypt, Mexico), written numbers usually resembled the Roman Numerals. Symbols were chosen for some numbers: "I" for one, "V" for five; "X" for ten, "L" for fifty, "C" for one hundred, "D" for five hundred, "M" for one thousand. The symbols were repeated as needed: "three" was "III" and "thirty" was "XXX".

We still see this notation in copyright and cornerstone dates: where **"1999" is "MDCCCCLXXXXVIIII"**. This is usually written as: **"MCMXCIX"**, where, when a smaller symbol precedes a larger one, subtract: **"IX" is "10 - 1" or "9"**.

Our modern (Arabic) notation is a positional, base-10 system, with ten numerals:

0, 1, 2, 3, 4, 5, 6, 7, 8, 9.

Beyond nine, numbers are represented as sums of multiples of powers of 10: Five thousand three hundred fourty-seven is $\mathbf{5{,}347 = 5\times10^3 + 3\times10^2 + 4\times10 + 7}$.

Notice the key role played by zero as a placeholder, as in:

One thousand one is $\mathbf{1{,}001 = 1\times10^3 + 0\times10^2 + 0\times10 + 1}$.

After the invention of zero as a placeholder, there was resistance to using zero alone to represent an actual number meaning "none". Post-Roman-Empire Europe invented the modern AD/BC calendar. (See "A Brief History of the Calendar") The year before 1AD was 1BC. There was no year zero, because zero was not an acceptable number to them. This is why the first millennium AD was 1AD to 1000AD. The last millennium BC was 1000BC to 1BC. The 21st century and third millennium AD did not begin until Jan. 1, 2001 (contrary to the many parties held on Dec. 31, 1999).

Basic Arithmetic Operations on Whole Numbers

Notice that our modern positional notation uses three arithmetic operations – addition, multiplication, and exponentiation – each of which may be defined for whole numbers in terms of the definition of them as "what you count with".

What is addition? Repeated counting. What is 27 + 54? Count out 27; then, count out 54 more; and then, start at the beginning and count the combination: 27 + 54 = 81. Addition is repeated counting. A little thought will convince you that the order does not matter: 27 + 54 = 54 + 27. The Arabs also invented algebra, a technique where letters are used to represent arbitrary or unknown numbers. Using this convention, we can express the fact that, the order in which addition is done is irrelevant, by:

$\mathbf{a + b = b + a}$ **Commutative Law of Addition.**

Similarly: 27 + (54 + 19) = (27 + 54) + 19 = 100, or, in general:

a + (b + c) = (a + b) + c Associative Law of Addition.

What is multiplication? Repeated addition. What is 5×3? It is 3 “5’s” added together: 5 + 5 + 5 = 15. What is 3×5? It is 5 “3’s” added together: 3 + 3 + 3 + 3 + 3 = 15. As with addition, the order is irrelevant. Consider 5 “3’s” and 3 “5’s”:

```
* * *
* * *                * * * * *
* * *                * * * * *
* * *                * * * * *
* * *
```

Both are clearly the same rectangle, 3 by 5 or 5 by 3, after rotating through 90°.

In general, with a little thought, we can see that:

a×b = b×a Commutative Law of Multiplication.

a×(b×c) = (a×b)×c Associative Law of Multiplication.

Combining “+” and “×”, consider: 3×(5 + 4) = 27 and (3×5) + (3×4) = 27:

```
***** + ****                 *****        ****
***** + ****      and        *****   +    ****
***** + ****                 *****        ****
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Again, it is the same pattern; so, in general, with a little thought:

a×(b + c) = a×b + a×c Distributive Law of Multiplication (over +).

What is exponentiation? Repeated multiplication. What is 5^3 = 5^3 ? It is 3 “5’s” multiplied together: 5×5×5 = 125. On the other hand, 3^5 = 3^5 = 3×3×3×3×3 = 243. The order matters! Exponentiation is NOT commutative. Because of this, the two roles have names; in $\mathbf{a^b}$, **a** is the base and **b** is the exponent. Sometimes, $\mathbf{a^b}$ is called “**a** raised to the power **b**”. Also, exponentiation is NOT associative:

2^(3^2) = 2^9 = 2^9 = 512 but (2^3)^2 = 8^2 = 8^2 = 64.

Exponentiation does obey some laws:

$2^3 \times 2^4 = (2\times2\times2)\times(2\times2\times2\times2) = 2^7$;

$(2^3)^4 = (2\times2\times2)\times(2\times2\times2)\times(2\times2\times2)\times(2\times2\times2) = 2^{12}$;

$(2\times3)^4 = (2\times3)\times(2\times3)\times(2\times3)\times(2\times3) = (2\times2\times2\times2)\times(3\times3\times3\times3) = 2^4\times3^4$.

So, with a little thought, we have the three laws of exponents:

$a^n \times a^m = a^{n+m}$	**(Add the exponents)**
$(a^n)^m = a^{nm}$	**(Multiply the exponents)**
$(a \times b)^n = a^n \times b^n$	**Distributive Law of Exponentiation (over ×).**

These 8 rules – 2 for "+", 3 for "×", 3 for "^" – are the basic laws of algebra.

But what about zero? Clearly: **a + 0 = 0 + a = a**, since zero means nothing and adding nothing does not change the count. In general:

$$a + 0 = 0 + a = a \quad \text{(zero is the additive identity).}$$

Aside: $a \times 1 = 1 \times a = a$ (one is the multiplicative identity)
$a \wedge 1 = a^1 = a$ (one is the exponential right-identity)

What is **a×0**? What is $a \wedge 0 = a^0$? At first thought, you might think both are zero. Since multiplication is repeated addition and exponentiation is repeated multiplication; and, if you repeat some operation zero times, you haven't done anything, so you have done nothing, and hence you have nothing. Right? No.

Recall two rules from above:

$$a \times (b + c) = a \times b + a \times c$$
$$a^n \times a^m = a^{n+m} \quad .$$

Whatever a×0 is, consider:

$$a \times b = a \times (b + 0) = a \times b + a \times 0.$$

The only way this can be true for all **a** and **b** is if **a×0 = 0**. What about **0×a**? Since multiplication is supposed to be commutative, we want **0×a = a×0 = 0**, which makes sense as adding zero to itself **a** times should give zero. Finally, **0×0 = 0**.

On the other hand, whatever $a \wedge 0 = a^0$ is, consider:

$$a^b = a^{b+0} = a^b \times a^0.$$

The only way this can be true for all non-zero **a** and **b** is if $a^0 = 1$. What about $0 \wedge a = 0^a$, for **a** not zero? Multiplying zero times itself any number of times still gives zero. So, for **a** not zero,

$$a \wedge 0 = a^0 = 1 \qquad \text{and} \qquad 0 \wedge a = 0^a = 0 \, .$$

Since exponentiation is not commutative, anyway, this is okay. But what about 0^0**?** Is it **0** or **1**? Sometimes, 0^0 is left undefined; but usually the convention $0^0 = 1$ is used, so that the function $f(x) = x^0 = 1$ is defined for all x.

Inverse Arithmetic Operations and More Numbers

Subtraction is the inverse operation to addition: 5 – 3 = 2 means 5 = 2 + 3. Subtraction "undoes" addition. Be careful - subtraction is neither commutative nor associative:

5 – 3 = 2	but	3 – 5 = ? (No Whole Number)
(8 – 5) – 2 = 1	but	8 – (5 – 2) = 5

Because we can't always subtract, negative numbers are invented: 3 – 5 = -2. Notice that '–' means subtraction while '-' means negative. The usual whole numbers (1,2,3,…) are called positive and the positive and negative whole numbers, together with zero (-0 = 0), are called the **Integers**.

With negative numbers, we see that subtraction is really just the same as addition: 5 – 3 = 5 + -3. In general:

a – a = a + -a = 0 **a – b = a + -b = -b + a = -(b – a)**

To keep the laws of algebra consistent, we must define:

a×(-b) = (-a)×b = -(a×b) **-a = (-1)×a** **-(-a) = a**.

What are these negative numbers? The Arabs who invented algebra seem to have realized that doing arithmetic with negative numbers was possible; but they thought it was somehow improper, that these negative numbers were not "really" numbers. Today, negative numbers have lost this stigma. From balancing bank overdrafts to recording cold temperatures and winning golf scores, the use of negative numbers is commonplace.

Division is the inverse operation to multiplication: 10 ÷ 2 = 5 means 10 = 5 × 2. Division "undoes" multiplication. Be careful - division is neither commutative nor associative:

24 ÷ 4 = 6	but	4 ÷ 24 = ? (No Integer)
(24 ÷ 4) ÷ 2 = 3	but	24 ÷ (4 ÷ 2) = 12

Because we can't always divide, fractions are invented: 4 ÷ 24 = $^4/_{24}$ = $^1/_6$.

In $^a/_b$, **a** is the **numerator** and **b** is the **denominator**. The number $^1/_a$ is the **reciprocal** of **a** and $^b/_a$ is the reciprocal of $^a/_b$. Common factors may be cancelled:

$$^{(a\times c)}/_{(b\times c)} = {^a}/_b.$$

With fractions, division is the same as multiplication: $24 \div 4 = 24\times(^1/_4) = {}^{24}/_4 = 6$.

To keep the laws of algebra consistent, we must define:

$$\mathbf{(^a/_b)\times(^c/_d) = {}^{(a\times c)}/_{(b\times d)}} \qquad \mathbf{{}^a/_a = 1} \qquad \mathbf{{}^{-a}/_{-b} = {}^a/_b}$$

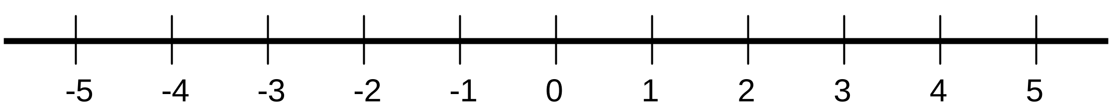
$$\mathbf{(^a/_b) \pm (^c/_d) = {}^{(a\times d\, \pm\, b\times c)}/_{(b\times d)}} \qquad \mathbf{{}^1/(^1/_a) = a} \qquad \mathbf{{}^{-a}/_b = {}^a/_{-b} = -{}^a/_b}$$

Notice that $0 \div a = {}^0/_a = 0$; since $0 = 0 \times a$. But, $a \div 0 = {}^a/_0$ = "division by zero" is impossible, since $a \div 0 = b$ must mean $a = b \times 0$, but $b \times 0 = 0$, which is not a. Zero cannot be a denominator in a fraction. What about ${}^0/_0$? It is undefined. The totality of all fractions are called the rational numbers.

What are fractions? How should we picture rational numbers? The ancient Egyptians originally invented fractions in order to measure distances more precisely. To measure distances, we first choose some basic unit of distance, like a pace, the length of a man's 2-step stride (a little over 5 feet). Using this standard, to find the distance between two points, you walk it and count out the number of paces. If it doesn't equal a whole number of paces, we divide a pace into some whole number of equal, smaller distances (say 15), and if the amount left over is 7 of these smaller distances, it is ${}^7/_{15}$ of a pace. (A fifteenth of a pace was called a "hand" and is still used to measure horses. A mile is the strange distance of 5280 feet because the Roman mile was originally 1000 paces.)

Using this notion of numbers as measurements of distance, we invent a geometrical picture of numbers as points on a line, with fractions filling in the points between the integral (± whole number) distances:

The Number Line



In terms of this picture, addition of positive numbers corresponds to a movement to the right of that distance; addition of negative numbers to a movement to the left. Multiplication by **a > 1** corresponds to a stretching of distance by that factor and division to a shrinking. Multiplication by **-1** corresponds to reflection about zero.

The Real Numbers

What about exponentiation and these new numbers – negatives and fractions? What is the inverse operation to exponentiation? The answers are related.

Two answers are easy.

What is $(-3)^4$? It's $(-3)\times(-3)\times(-3)\times(-3) = 81$ (using $(-a)(-b) = a\times b$).

What is $(1/3)^4$? It's $(1/3)\times(1/3)\times(1/3)\times(1/3) = 1/81$

The original definitions work.

The other answers are less clear. What is $(3)^{-4}$? What is $(3)^{1/4}$? What is something times itself a negative number of times? A fractional number of times? Again, the key is in the laws of algebra.

Whatever $\mathbf{a^{-n}}$ is, consider: $\mathbf{a^{-n} \times a^{n} = a^{-n+n} = a^{0} = 1}$; hence,

$\mathbf{a^{-n} = 1/a^{n}}$: negative exponents are just reciprocals;

There's nothing new here except another way of writing fractions.

Whatever $\mathbf{a^{1/n}}$ is, consider: $\mathbf{(a^{1/n})^{n} = a^{n/n} = a^{1} = a}$; hence, $\mathbf{a^{1/n}}$ is a number which, when raised to the n^{th} power, is a. This new operation is called "taking the n^{th} root". It is the inverse operation of rising to the n^{th} power:

$$\mathbf{a^{1/n} = {}^{n}\sqrt{a} = b \quad \text{means} \quad a = b^{n}}$$

where the symbol "√" is called a radical. Identifying roots with fractional exponents, we see from the laws of exponents that:

$$\mathbf{{}^{n}\sqrt{(a\times b)} = (a\times b)^{1/n} = (a^{1/n})\times(b^{1/n}) = ({}^{n}\sqrt{a})\times({}^{n}\sqrt{b})}$$

But is "taking roots" possible? Sometimes. Clearly, $8^{1/3} = {}^{3}\sqrt{8} = 2$, since $2^3 = 8$. Also, $(-8)^{1/3} = {}^{3}\sqrt{(-8)} = -2$, since $(-2)^3 = -8$. Similarly, $243^{1/5} = {}^{5}\sqrt{243} = 3$ and $128^{1/7} = {}^{7}\sqrt{128} = 2$. Odd roots seem okay. What about even roots? What is $4^{1/2} = {}^{2}\sqrt{4} = \sqrt{4} = ?$ (A radical without a leading number, "√", is called a "square root" and "${}^{2}\sqrt{}$" is implied; this is because a number raised to the power two, a^2, is often called "a-squared".) There seem to be two answers: 2 and -2, since $2^2 = 4$ and $(-2)^2 = 4$. For even roots, the positive root is always and solely used. Similarly, $16^{1/4} = {}^{4}\sqrt{16} = 2$ and $81^{1/4} = {}^{4}\sqrt{81} = 3$.

What about $(-4)^{1/2} = {}^{2}\sqrt{(-4)} = \sqrt{(-4)} = ?$ Since positive times positive is positive, and negative times negative is positive, it would seem that it is not "really" possible to find the square root of a negative number – and this is "really" true – it is not "really" possible to take an even root of a negative number.

But let's look at the "square root" operation a little more closely. What is **$\sqrt{2}$**? Is it a fraction? $(^3/_2)^2 = {}^9/_4 = 2.25$. So, $^3/_2$ is close to the $\sqrt{2}$. $(^{17}/_{12})^2 = {}^{289}/_{144} \approx 2.007$. So, $^{17}/_{12}$ is even closer to the $\sqrt{2}$. Is there some fraction which is exactly $\sqrt{2}$? The answer is no. Similarly, there is no fraction which is exactly $\sqrt[3]{2}$. We need to invent a new type of number in order to take roots of all but a few numbers.

These new numbers are an extension of our usual Arabic numbers to include negative powers of 10:

$$\mathbf{5{,}348.125 = 5\times10^3 + 3\times10^2 + 4\times10^1 + 8\times10^0 + 1\times10^{-1} + 2\times10^{-2} + 5\times10^{-3}}$$
$$\mathbf{5\times10^3 + 3\times10^2 + 4\times10^1 + 8 + 1\times(^1/_{10}) + 2\times(^1/_{100}) + 5\times(^1/_{1000})}$$

This is the notation for writing fractions as decimal expansions. It was first used in Renaissance Europe. Some fractions have decimal expansions that end:

$\mathbf{^1/_2 = 0.5}$ $\mathbf{^1/_4 = 0.25}$ $\mathbf{^1/_{40} = 0.025}$

Other fractions have decimal expansions that repeat endlessly:

$\mathbf{^1/_3 = 0.333333\ldots}$ $\mathbf{^1/_6 = 0.166666\ldots}$ $\mathbf{^1/_{11} = 0.090909}\ldots$

All rational numbers have decimal expansions which either end or repeat endlessly. But a number like $\sqrt{2}$ has a decimal expansion which neither ends nor repeats:

$$\mathbf{\sqrt{2} = 1.4142135623730950488016887242097\ldots}$$

Numbers like **$\sqrt{2}$** are called **irrational**, meaning "not rational" (and not meaning crazy). When actually calculating with irrational numbers, one generally "cuts off" the expansion at some point and uses a rational approximation: $\sqrt{2} \approx 1.4142$, with as many decimal places as is required for the accuracy desired.

The totality of all the rational and irrational numbers are called the real numbers. By including irrational numbers, one can take odd roots ($\mathbf{\sqrt[n]{a} = a^{1/n}}$ for n odd) for any real number and even roots ($\mathbf{\sqrt[n]{a} = a^{1/n}}$ for n even) for any positive real number.

What about other types of exponents besides roots? What is $\mathbf{a^{n/m}}$, where the exponent is a fraction? This is simple – we use the law of exponents, $\mathbf{(a^n)^m = a^{nm}}$, and define:

$$\mathbf{a^{n/m} = (a^n)^{1/m} = \sqrt[m]{(a^n)} = (\sqrt[m]{a})^n}$$

What is $\mathbf{a^{\sqrt{2}}}$? What is $\mathbf{a^b}$, for **b** irrational? One may calculate it as accurately as is needed by using a rational approximation to the irrational number. Because negative numbers don't have even roots, fractional and irrational exponents are usually allowed only for **$a>0$**.

The invention of the real numbers completes the picture of the rational numbers as points on the number line. With rational numbers only, the line appears to be solid, but there are actually holes in it. Certain distances, like $\sqrt{2}$, are not representable as rational numbers. Real numbers are needed to represent all possible distances. The length of the hypotenuse of a right isosceles triangle with two sides of length 1 is $\sqrt{2}$.

More Operations and Numbers?

Under the six Arithmetic operations:

$$\mathbf{a + b \quad a - b \quad a \times b \quad a \div b \quad a^b = a\text{^}b \quad \sqrt[a]{b} = b\text{^}({}^1/_a)}$$

the real numbers obey the Laws of Algebra (see appendix), although some operations, like dividing by zero and taking even roots of negative numbers, are not possible.

Are there any other types of numbers? Can we invent a new type of number to allow even roots of negative numbers? Are there any other arithmetic operations? The answers to all are “yes”.

Let’s review the inverse operations again:

$5 = 2 + 3$	means	$5 - 2 = 3$	and	$5 - 3 = 2$
$6 = 2 \times 3$	means	$6 \div 2 = 3$	and	$6 \div 3 = 2$
$8 = 2\text{^}3 = 2^3$	means	$\sqrt[3]{8} = 2$	and	$\sqrt[2]{8}$ = ?Oops!

The square root of 8 is not 3! The square root of 8 is $\sqrt{8} = 2.828427124\ldots$

Unlike addition and multiplication, where subtraction and division are inverse operations for either side, taking roots is only a one-sided (right-sided) inverse for exponentiation. This is because exponentiation is NOT commutative. What is the left-sided inverse? What operation performed on 8 and 2 yields 3? We define logarithms as:

$$\mathbf{c = \log_a(b) \quad \text{means} \quad b = a^c}$$

The “$\mathbf{\log_a(b)}$” is that exponent **c** to which **a** must be raised to yield **b**.

Notice that:

$\log_2(8) = 3$ because $8 = 2^3$

$\log_3(81) = 4$ because $81 = 3^4$

$\log_{10}(^1/_{100}) = -2$ because $10^{-2} = ^1/_{100}$.

Since irrational exponents are only defined for positive bases and a positive base to any power is positive, we define $\mathbf{\log_a(b)}$ only for **a>0** and **b>0**. Logarithms are often not included as a basic arithmetic operation, but they logically should be.

Reversing the Laws of Exponents gives the Laws of Logarithms:

$$\log_a(1) = 0 \qquad \log_a(a^c) = c \qquad \log_c(b) = \log_a(b) / \log_a(c)$$

$$\log_a(b) + \log_a(c) = \log_a(bc) \qquad \log_a(b^c) = c \log_a(b)$$

Finally, let's reconsider our rule that even roots of negative numbers do not exist. What if we just invent a number, call it **i**, with: $\mathbf{i^2 = -1}$?

$$i^2 = -1 \qquad \sqrt[2]{(-1)} = \sqrt{(-1)} = i$$

With this done, what is $\sqrt{(-4)}$? Using the rules of algebra, it must be:

$\sqrt{(-4)} = \sqrt{(4\times(-1))} = \sqrt{4} \times \sqrt{(-1)} = 2 \times i = 2i$.

In fact, with a little more work, it turns out we can take the square root of i:

$\sqrt{i} = 1/\sqrt{2} + i/\sqrt{2}$ or $\sqrt{i} = -1/\sqrt{2} + -i/\sqrt{2}$

What does it mean to add a real number to i? To multiply i by a real number? Don't worry what it means, just do it! If we do, we find that the laws of algebra still work and we can perform every possible operation on every type of number. Yes, we can even take roots and logs of complex numbers, although discussing this requires some advanced mathematics. There are only two exceptions: **division by zero and logs of zero are still banned**.

The number **i** is called the **imaginary unit**.

Numbers like $\mathbf{ai = a\times i}$ where **a** is a real number are called **imaginary numbers**. There is no "real" $\sqrt{(-4)}$; but there is an "imaginary" $\sqrt{(-4)} = 2i$. Finally, numbers like **a + bi** where **a** and **b** are real numbers are called **complex numbers**. The square root of i, $\sqrt{i}$, is a complex number.

This is the end of the story. Beginning with the counting numbers, we have successively invented:

> **negative numbers** so we can always subtract,
> **rational numbers** so we can always divide (except zero),
> **real numbers** so we can always take roots and logs (of positives),
> **complex numbers** so we can always take roots and logs (except zero);

and this is the end of the story. We cannot invent any new types of numbers and still have the laws of algebra hold true. While additional mathematical concepts may be invented which behave something like numbers, matrices for instance, they do not obey all the laws of algebra. In particular, matrices violate the commutative law of multiplication:

$$\mathbf{a}\times\mathbf{b} \neq \mathbf{b}\times\mathbf{a}.$$

The story of numbers, that great creation of the human imagination, ends with the invention of imaginary numbers. It's almost ironic.

Laws of Algebra

Operations: where a, b, c, d are (whole) numbers

Addition: $a + b$ (Repeated Counting)

- **Commutative: $a + b = b + a$** — **$a + 0 = a$**
- **Associative: $a + (b + c) = (a + b) + c$**

Multiplication (Repeated addition): $a \times b = a \cdot b = a*b = ab$

- **Commutative: $ab = ba$** — **$a \times 0 = 0$**
- **Associative: $a(bc) = (ab)c$**
- **Distributive: $a(b + c) = ab + ac = (b + c)a$**

Exponentiation (Repeated multiplication): $a^b = a\hat{}b$

- **Where the base $a > 0$ or the exponent b is an integer**
- **Neither commutative nor associative**
  - **$a^b a^c = a^b \times a^c = a^{b+c}$** — **$a^0 = 1$**
  - **$(a^b)^c = a^{bc}$**
  - **$(ab)^c = a^c b^c$**

Subtraction (Inverse addition): $a - b$

- **Neither commutative nor associative**
- **Invent Negatives: $a + -a = 0$**
  - **$-(-a) = a$**
  - **$a + (-b) = a - b = -(b - a)$**
  - **$(-1)a = -a$**

Division (Inverse multiplication): $a \div b = a/b$ (where $b \neq 0$)

- **Neither commutative nor associative**
- **Invent Fractions: $a \times (1/a) = 1$**
  - **$a/b = a \times (1/b)$**
  - **$(a/b)(c/d) = (ac)/(bd)$**
  - **$(a/b) + (c/d) = (ad + bc)/(bd)$**
- **Negative Exponents: $a^{-b} = 1/a^b = (1/a)^b$**

Roots and Logarithms (Inverse exponentiation) (where base > 0)

- **Invent Reals as infinite decimal expansions**
- **Roots: $(\sqrt[b]{a})^b = \sqrt[b]{(a^b)} = a$**
- **Fractional Exponents: $(a^{1/b}) = \sqrt[b]{a}$**
- **Logarithms: $b\hat{}(\log_b a) = a = \log_b(b^a)$**

Precedence Rules:

- **First: Do Exponents, Roots and Logarithms, left to right;**
- **Second: Do Multiplications and Divisions, left to right;**
- **Third: Do Additions and Subtractions, left to right.**
- **Parentheses override this normal order.**

Algorithms for Doing Arithmetic

Back in grade school, when we learned to do arithmetic, many of us were told that addition WAS that process we learned:

$$\begin{array}{r} 388 \\ \underline{+294} \\ 682 \end{array}$$

Remember: “Eight plus four is twelve, that’s two, carry one; one plus eight plus nine is eighteen, that’s eight, carry one; one plus three plus two is six.” This is not really true. Addition is repeated counting. That process we learned is a short cut for doing addition more quickly. Imagine how long it would take to count out 388, and then count out 294 more, and then recount out 682!

Similarly, subtraction and multiplication and division are really what they were defined to be above. Those processes we learned in order to do subtraction and multiplication and division are shortcuts. Imagine how long it would take to add 388 to itself 294 times! But it is relatively quick and easy to compute the product, $388 \times 294 = 114{,}072$, by long-hand multiplication – and even easier with a calculator.

Such shortcuts are called algorithms and were invented by the Arabs as part of their epic invention of the algebra. It is an interesting exercise in logic to justify each of the algorithms for arithmetic in terms of the laws of algebra and meaning of Arabic numerals as sums of products of powers of 10.

Recommended Reading

Dantzig, Tobias; "Number - The Language of Science". An excellent presentation on the history of numbers and number systems, including the achievements of Chinese and Mayan cultures.

Hogben, Lancelot; "Mathematics for the Million". The classic popularization .

Seife, Charles; "Zero: The Biography of a Dangerous Idea"
A very original history of a single number.

van der Waerden, B.L.; "Science Awakening: Egyptian, Babylonian and Greek Mathematics". The classic treatment of the mathematics of these ancient cultures, with all the nitty-gritty details.

Singh, Jagjit; "Great Ideas of Modern Mathematics". Now out of print, but still very much worth reading.